Interpretations of some parameter dependent generalizations of classical matrix ensembles

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Two types of parameter dependent generalizations of classical matrix ensembles are defined by their probability density functions (PDFs). As the parameter is varied, one interpolates between the eigenvalue PDF for the superposition of two classical ensembles with orthogonal symmetry and the eigenvalue PDF for a single classical ensemble with unitary symmetry, while the other interpolates between a classical ensemble with orthogonal symmetry and a classical ensemble with symplectic symmetry. We give interpretations of these PDFs in terms of probabilities associated to the continuous Robinson-Schensted-Knuth correspondence between matrices, with entries chosen from certain exponential distributions, and non-intersecting lattice paths, and in the course of this probability measures on partitions and pairs of partitions are identified. The latter are generalized by using Macdonald polynomial theory, and a particular continuum limit — the Jacobi limit — of the resulting measures is shown to give PDFs related to those appearing in the work of Anderson on the Selberg integral. By interpreting Anderson's work as giving the PDF for the zeros of a certain rational function, it is then possible to identify random matrices whose eigenvalue PDFs realize the original parameter dependent PDFs. This line of theory allows sampling of the original parameter dependent PDFs, their Anderson-type generalizations and associated marginal distributions, from the zeros of certain polynomials defined in terms of random three term recurrences.

1 Introduction

This paper is a companion to our work [19]. In [19] we studied the correlation functions for the probability density functions (PDFs)

$$\frac{1}{C}\prod_{j=1}^{2n} e^{-x_j/2}\prod_{j=1}^{n} e^{A(x_{2j-1}-x_{2j})/2}\prod_{1\le j<k\le 2n}(x_j-x_k) \tag{1.1}$$

$$\frac{1}{C}\prod_{j=1}^{2n} e^{-x_j/2}\prod_{j=1}^{n} e^{A(x_{2j-1}-x_{2j})/2}\prod_{1\le j<k\le n}(x_{2j-1}-x_{2k-1})(x_{2k}-x_{2j}), \tag{1.2}$$

where C is the normalization (throughout C will be used to denote *some* normalization; we remark too that it is required $A<1$ for (1.1) to be normalizable) and

$$x_1 > x_2 > \cdots > x_{2n} \ge 0. \tag{1.3}$$

The first of these was isolated [4] in the context of a study of generalizations of Ulam's problem — the computation of the distribution of the length of the longest increasing subsequence in a random permutation. The second, although not given explicitly in the same paper that (1.1) was noted, has its origin in a particular model introduced in [4].

In [19] we also studied the correlation functions for the particular parameter dependent PDFs

$$\frac{1}{C}\prod_{j=1}^{2n} x_j^{(a-1)/2}\prod_{l=1}^{n}\Big(\frac{x_{2l}}{x_{2l-1}}\Big)^{-A/2}\prod_{1\le j<k\le 2n}(x_j-x_k), \tag{1.4}$$

$$\frac{1}{C}\prod_{j=1}^{2n} x_j^{(a-1)/2}\prod_{l=1}^{n}\Big(\frac{x_{2l}}{x_{2l-1}}\Big)^{-A/2}\prod_{1\le j<k\le n}(x_{2j-1}-x_{2k-1})(x_{2j}-x_{2k}), \tag{1.5}$$

where the condition $A<a+1$ is required for (1.4) and (1.5) to be normalizable and

$$1>x_1>x_2>\cdots>x_{2n}>0.$$

(In [19] these PDFs were defined on $(-1,1)$ rather than $(0,1)$ as done here; to define the former simply change variables $x_j\mapsto(1-x_{2n+1-j})/2$ in the above.) The PDFs (1.4) and (1.5) were identified in [19] as the only parameter dependent extensions of classical matrix ensembles with an even number of eigenvalues, in addition to (1.1) and (1.2), with the special property that after integrating over every second eigenvalue the eigenvalue PDF of a matrix ensemble with symplectic and unitary symmetry respectively results. As noted in [19], by scaling the variables and parameters

$$x_j\mapsto x_j/L,\quad a\mapsto L,\quad A\mapsto LA \tag{1.6}$$

and taking the limit $L\to\infty$, (1.4) and (1.5) reduce to (1.1) and (1.2) respectively.

Let us remark at this point how the above PDFs relate to the classical matrix ensembles. Following the notation of [18], we specify a matrix ensemble with orthogonal ($\beta=1$), unitary ($\beta=2$) or symplectic ($\beta=4$) symmetry by the eigenvalue PDFs

$$\frac{1}{C}\prod_{l=1}^{N} g(x_l)\prod_{1\le j<k\le N}|x_k-x_j|^{\beta}. \tag{1.7}$$

A classical matrix ensemble then refers to an eigenvalue PDF of the form (1.7) with $g(x)$ a classical weight function — Gaussian (e^{-x^2}), Laguerre ($x^a e^{-x}$), Jacobi ($x^a(1-x)^b$) or Cauchy ($(1+x^2)^{-\alpha}$). Thus we recognize (1.1) in the case $A=0$ as the Laguerre orthogonal ensemble with $a=0$ and $2n$ eigenvalues which we denote as $\mathrm{LOE}_{2n}|_{a=0}$, while we recognize (1.4) in the case $A=0$ as the Jacobi orthogonal ensemble with $a\mapsto(a-1)/2$, $b=0$ and $2n$ eigenvalues which we denote as $\mathrm{JOE}_{2n}|_{\substack{a\mapsto(a-1)/2\\ b=0}}$. In the limit $A\to-\infty$ of (1.1), (1.4) each pair of coordinates collapses, and after appropriate rescaling and renaming of the collapsed pairs the limiting PDFs are

$$\frac{1}{C}\prod_{j=1}^{n} e^{-x_j}\prod_{1\le j<k\le n}(x_j-x_k)^4, \tag{1.8}$$

$$\frac{1}{C}\prod_{j=1}^{n} x_j^{a+1}\prod_{1\le j<k\le n}(x_j-x_k)^4. \tag{1.9}$$

The first of these is the Laguerre symplectic ensemble with parameter $a=0$, denoted as $\mathrm{LSE}_n|_{a=0}$, while the second is the Jacobi symplectic ensemble with $a\mapsto a+1$, $b=0$, denoted as $\mathrm{JSE}_n|_{\substack{a\mapsto a+1\\ b=0}}$. Consequently we have that (1.1) and (1.4) interpolate between particular orthogonal ensembles and symplectic ensembles. Regarding the PDFs (1.2) and (1.5), we require the fact [18] that superimposing two orthogonal ensembles as specified by (1.7) with $\beta=1$ at random gives the eigenvalue PDF

$$\frac{1}{C}\prod_{j=1}^{2n} g(x_j)\prod_{1\le j<k\le n}(x_{2j-1}-x_{2k-1})(x_{2k}-x_{2j}). \tag{1.10}$$

Thus with $A = 0$, (1.2) and (1.5) are recognized as the superimposed ensembles $\mathrm{LOE}_n|_{a=0} \cup \mathrm{LOE}_n|_{a=0}$ and $\mathrm{JOE}_n|_{\substack{a\mapsto(a-1)/2\\ b=0}} \cup \mathrm{JOE}_n|_{\substack{a\mapsto(a-1)/2\\ b=0}}$ respectively. In the $A \to -\infty$ limit (1.2) and (1.5) effectively reduce to [19]

$$\frac{1}{C}\prod_{j=1}^n e^{-x_j} \prod_{1\le j<k\le n} (x_k - x_j)^2 \tag{1.11}$$

$$\frac{1}{C}\prod_{j=1}^n x_j^a \prod_{1\le j<k\le n} (x_k - x_j)^2 \tag{1.12}$$

respectively, which specify the Laguerre unitary ensemble with $a = 0$, denoted $\mathrm{LUE}_n|_{a=0}$, and the Jacobi unitary ensemble with $b = 0$, denoted $\mathrm{JUE}_n|_{b=0}$. Consequently (1.2) and (1.5) interpolate between particular superimposed orthogonal ensembles and unitary ensembles.

Our objective in this paper is to give interpretations of each of (1.1), (1.2), (1.4) and (1.5) as probability densities relating to longest increasing subsequence problems in settings analogous to that already known for (1.1), and also to specify parameter dependent random matrices which have these distributions as their eigenvalue PDF. We will see that these two pursuits are intimately related via a conditional PDF to be referred to as the Anderson density. To establish this link requires first generalizing the most immediate interpretation of the PDFs in the setting of longest increasing subsequence problems/ last passage percolation, discussed in Section 2, to a measure on pairs of partitions λ, κ with λ/κ a horizontal strip suggested by Macdonald polynomial theory. This is done in Section 3. Taking a particular continuum limit, already known from Section 2 as the Jacobi limit, gives rise to the Anderson density. The crucial point in making the link with random matrix theory is an interpretation of the workings of Anderson's paper [2] as giving the density of zeros of a certain random rational function. This same random rational function occurs as part of the characteristic equation for the random projection of a fixed matrix with in general degenerate eigenvalues. Such a random projection is used in Section 4.1 to give the construction of random matrices with eigenvalue PDFs (1.4) and (1.5). It is shown in Section 4.2 that the intricacies of the relationship between the Anderson density and the random rational function allow a generalization of the joint densities (1.4) and (1.5) — as well as an associated marginal density — to be sampled from the zeros of a polynomial generated by a random three term recurrence.

The Anderson density and the corresponding random rational function have a well defined Laguerre limit giving rise to a generalization of the joint densities (1.1) and (1.2). As noted in Section 5.1 the Laguerre limit of the random rational function occurs as part of the eigenvalue equation for a random rank 1 projection of a fixed matrix with in general degenerate eigenvalues, allowing for the construction of random matrices with eigenvalue PDFs (1.1) and (1.2). Furthermore it is shown in Section 5.2 that the Laguerre limit of the random three term recurrences of Section 4.2 generates polynomials from which a generalization of (1.1) and (1.2), and an associated marginal density corresponding to the Laguerre limit of the Selberg integral, can be sampled.

In Section 6 we carry out a further limiting analysis of the results of Section 4, this time characterized by the Jacobi type weights associated to the Anderson density degenerating to Gaussian weights. A special case of the resulting parameter dependent Gaussian ensemble is known from Section 2.3 as the joint probability density for a particular last passage percolation model.

In the course of our study we encounter the need to further develop/ reformulate some existing theory, in particular the Robinson-Schensted-Knuth correspondence. We also encounter some consequences of our findings by way of new insights into some existing results. Such points are presented in the Appendices.

2 Last passage percolation and tableaux coordinates

The generalizations of Ulam's problem of interest to us can in turn be regarded as generalizations of a last passage percolation model introduced by Johansson [23]. To define the latter consider the right quadrant square lattice $\{(i,j) : i,j \in \mathbb{Z}^+\}$. Associate with each lattice site (i,j) a random non-negative integer variable $x_{i,j}$, chosen from the geometric distribution with parameter $a_i b_j$ so that

$$\Pr(x_{i,j} = k) = (1 - a_i b_j)(a_i b_j)^k. \tag{2.1}$$

For given non-negative parameters $a_1, a_2, \dots, b_1, b_2, \dots$ the quantity of interest is the distribution of the so-called last passage time

$$L(n_1, n_2) := \max \sum_{(1,1)\mathrm{u/rh}(n_1,n_2)} x_{i,j} \tag{2.2}$$

where the notation $(1,1)\mathrm{u/rh}(n_1,n_2)$ denotes that the sum is over all lattice points in a path starting at $(1,1)$ and finishing at (n_1,n_2), with segments which are either up or right horizontal (such a path is said to be weakly increasing). The celebrated Robinson-Schensted-Knuth (RSK) correspondence (see e.g. [20, 36]) gives a bijection between $n_1 \times n_2$ non-negative integer matrices and pairs of semi-standard tableaux of the same shape $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ say with the crucial feature that $\mu_1 = L(n_1, n_2)$. It is these variables which after rescaling give rise to the parameter dependent PDFs listed in the Introduction.

We require some (mostly) known facts about the RSK correspondence in the situation that there is a measure on the space of non-negative integer matrices as implied by (2.1). First, the probability an $n_1 \times n_2$ non-negative integer matrix with this measure corresponds to a pair of semi-standard tableaux with shape μ, one of content n_1, the other of content n_2 is given by [26]

$$\prod_{i=1}^{n_1} \prod_{j=1}^{n_2} (1 - a_i b_j) s_\mu(a_1, \dots, a_{n_1}) s_\mu(b_1, \dots, b_{n_2}), \tag{2.3}$$

where s_μ denotes the Schur polynomial. Second, for $n_2 \ge n_1$, the joint probability that an $n_1 \times (n_2 + 1)$ non-negative integer matrix with measure implied by (2.1) corresponds to a pair of semi-standard tableaux with shape μ, content n_1 and $n_2 + 1$, and that the $n_1 \times n_2$ bottom left sub-block corresponds to a pair of semi-standard tableaux with shape κ, content n_1 and n_2 is

$$\prod_{i=1}^{n_1} \prod_{j=1}^{n_2+1} (1 - a_i b_j) s_\mu(a_1, \dots, a_{n_1}) s_\kappa(b_1, \dots, b_{n_2}) b_{n_2+1}^{\sum_{j=1}^{n_1} \mu_j - \kappa_j} \tag{2.4}$$

where

$$\mu_1 \ge \kappa_1 \ge \mu_2 \ge \kappa_2 \ge \cdots \ge \mu_{n_1} \ge \kappa_{n_1} \ge 0. \tag{2.5}$$

Note that by the assumption $n_1 \le n_2$, we require $\ell(\mu)$ — the number of non-zero parts of μ — to be less than or equal to n_1 for (2.4) to be non-zero. If instead $n_1 > n_2$, then (2.4) holds with

$$b_{n_2+1}^{\sum_{j=1}^{n_1} \mu_j - \kappa_j} \mapsto b_{n_2+1}^{\sum_{j=1}^{n_2} (\mu_j - \kappa_j) + \mu_{n_2+1}} \tag{2.6}$$

and (2.5) must be modified to read

$$\mu_1 \ge \kappa_1 \ge \mu_2 \ge \kappa_2 \ge \cdots \ge \mu_{n_2} \ge \kappa_{n_2} \ge \mu_{n_2+1} \ge 0. \tag{2.7}$$

Third, in the case of matrices symmetric about $i = j$ (here i denotes the row counted from the bottom), when the RSK correspondence maps the matrix to a single semi-standard tableau, with the parameters of the geometric distribution chosen so that

$$\Pr(x_{i,j} = k) = (1 - a_i a_j)(a_i a_j)^k, \ \ i < j \qquad \Pr(x_{i,i} = k) = (1 - a_i) a_i^k, \tag{2.8}$$

the derivation of (2.3) given in [26] implies the probability that the tableau has shape μ is given by

$$\prod_{i=1}^{n}(1-a_i)\prod_{1\le i<j\le n}(1-a_ia_j)s_\mu(a_1,\dots,a_n). \tag{2.9}$$

Fourth, in this latter situation, the correspondence is such that

$$\sum_{j=1}^{n}x_{j,j}=\sum_{j=1}^{n}(-1)^{j-1}\mu_j. \tag{2.10}$$

In [26], (2.10) is given in the form

$$\sum_{j=1}^{n}x_{j,j}=\sum_{j=1}^{\ell(\mu)}\frac{1}{2}(1-(-1)^{\mu'_j}) \tag{2.11}$$

where μ'_j denotes the length of the jth column of the diagram of μ, which is the sum of the number of odd columns; simple reasoning shows that (2.11) is equivalent to (2.10). In Appendix A we will give a self contained derivation of (2.10) which is in keeping with the interpretation of the RSK correspondence as a cascade of growth models given in [24]. The same ideas will be used to derive (2.4).

It follows from (2.9) and (2.10) that if (2.8) is modified so that the second equation reads

$$\Pr(x_{i,i}=k)=(1-\alpha a_i)(\alpha a_i)^k \tag{2.12}$$

then the probability that the tableau has shape μ is given by

$$\prod_{i=1}^{n}(1-\alpha a_i)\prod_{1\le i<j\le n}(1-a_ia_j)\alpha^{\sum_{j=1}^{n}(-1)^{j-1}\mu_j}s_\mu(a_1,\dots,a_n). \tag{2.13}$$

We will show that (1.1) and (1.4) result from this probability. The PDFs (1.2) and (1.5) will be derived from (2.4).

Before undertaking the derivations, we note that from the origin of (2.4) as a joint probability with the corresponding marginal density given or implied by (2.3), the former must satisfy special identities with respect to summation over κ and summation over μ. Thus let R denote the region (2.5) in the case $n_2\ge n_1$, and the region (2.7) in the case $n_1>n_2$. Then since summing (2.4) over $\kappa\in R$ must give (2.3) with $n_2\mapsto n_2+1$, it follows

$$\sum_{\kappa:\kappa\in R}s_\kappa(b_1,\dots,b_{n_2})b_{n_2+1}^{|\mu|-|\kappa|}=s_\mu(b_1,\dots,b_{n_2+1}) \tag{2.14}$$

where $|\mu|=\sum_{i=1}^{\ell(\mu)}\mu_i$ and similarly the meaning of $|\kappa|$. In fact (2.14) is a well known recurrence satisfied by the Schur polynomials [28, special case of (5.10) pg. 72]. Similarly, summing (2.4) over μ must give (2.3) with $\mu\mapsto\kappa$ and so

$$\prod_{i=1}^{n_1}(1-a_ib_{n_2+1})\sum_{\mu:\mu\in R}s_\mu(a_1,\dots,a_{n_1})b_{n_2+1}^{|\mu|-|\kappa|}=s_\kappa(a_1,\dots,a_{n_1}) \tag{2.15}$$

which is also a known Schur polynomial identity [28, special case of (1) pg. 93].

2.1 Jacobi limit

The parameter dependent Jacobi ensembles (1.4), (1.5) are obtained from (2.13), (2.4) respectively by specializing the parameters $\{a_i\}$, $\{b_j\}$. In (2.13) we choose

$$(a_1,\dots,a_n)=(z,zt,zt^2,\dots,zt^{n-1}) \tag{2.16}$$

while in (2.4) we choose

$$(a_1,\dots,a_{n_1}) = (z, zt, zt^2, \dots, zt^{n_1-1}), \qquad (b_1,\dots,b_{n_2}) = (z, zt, zt^2, \dots, zt^{n_2-1}). \tag{2.17}$$

The probabilities then assume an explicit form in terms of the parts of μ and κ due to the evaluation formula [28]

$$\begin{aligned} s_\lambda(1,t,\dots,t^{n-1}) &= t^{\sum_{i=1}^n (i-1)\lambda_i} \prod_{1\le i<j\le n} \frac{1-t^{\lambda_i-\lambda_j-i+j}}{1-t^{j-i}} \\ &= \frac{t^{-\sum_{j=1}^n (j-1)(n^*-j)}}{\prod_{l=1}^{n-1}(t;t)_l} \prod_{1\le i<j\le n} (t^{h_j} - t^{h_i}) \end{aligned} \tag{2.18}$$

where in the second line, which follows from the first by simple manipulation, $(t;t)_l := (1-t)(1-t^2)\cdots(1-t^l)$, $h_j := \lambda_j + n^* - j$ and n^* is arbitrary.

Substituting (2.18) with $n^* = n$ in (2.13) we deduce the following result.

Proposition 1. *On each site (i,j) of the $n\times n$ square lattice, specify a non-negative integer $x_{i,j}$ according to the probability distribution*

$$\begin{aligned} \Pr(x_{i,j} = k) &= (1 - z^2 t^{i+j-2})(z^2 t^{i+j-2})^k \quad i<j \\ \Pr(x_{i,i} = k) &= (1 - \alpha z t^{i-1})(\alpha z t^{i-1})^k \end{aligned}$$

and impose the symmetry constraint that $x_{i,j} = x_{j,i}$ for $i > j$. Then the probability that a configuration $[x_{i,j}]$ gives a tableau of shape μ under the RSK correspondence is equal to

$$c_n(z,\alpha,t) z^{\sum_{j=1}^n h_j} \alpha^{\sum_{j=1}^n (-1)^{j-1} h_j} \prod_{1\le i<j\le n} (t^{h_j} - t^{h_i}), \tag{2.19}$$

$$\begin{aligned} c_n(z,\alpha,t) &:= z^{-\sum_{j=1}^n (n-j)} \alpha^{-[n/2]} \frac{t^{-\sum_{j=1}^n (j-1)(n-j)}}{\prod_{l=1}^{n-1}(t;t)_l} \\ &\quad \times \prod_{i=1}^n (1-\alpha z t^{i-1}) \prod_{1\le i<j\le n} (1 - z^2 t^{i+j-2}), \end{aligned} \tag{2.20}$$

where $h_j := \mu_j + n - j$ and thus $h_1 > h_2 > \cdots > h_n \ge 0$. Furthermore in the scaled (Jacobi) limit

$$t = e^{-1/L},\ \ z = e^{-a/L},\ \ \alpha = e^{-a_1/L},\ \ h_j/L = x_j,\ \ L\to\infty, \tag{2.21}$$

when each lattice site (i,j) specifies a non-negative continuous exponential random variable with site dependent variance

$$\begin{aligned} \Pr(x_{i,j} \in [y, y+dy]) &= (i+j-2+2a) e^{-y(i+j-2+2a)} dy, \qquad i<j \\ \Pr(x_{i,i} \in [y, y+dy]) &= (i-1+a+a_1) e^{-y(i-1+a+a_1)} dy, \end{aligned} \tag{2.22}$$

the probability (2.19) multiplied by L^n tends to the PDF

$$\tilde c_n(a,a_1) e^{-a\sum_{j=1}^n x_j} e^{-a_1 \sum_{j=1}^n (-1)^{j-1} x_j} \prod_{1\le i<j\le n} (e^{-x_j} - e^{-x_i}), \tag{2.23}$$

$$\tilde c_n(a,a_1) := \frac{\Gamma(a+a_1+n)}{\Gamma(a+a_1)} \frac{1}{\prod_{l=1}^{n-1} l!} \prod_{i=1}^{n-1} \frac{\Gamma(2a+i+n-1)}{\Gamma(2a+2i-1)} \tag{2.24}$$

where $x_1 > x_2 > \cdots > x_n > 0$.

After the change of variables and replacement of parameters

$$e^{-x_j} \mapsto x_{n+1-j}, \;\; a \mapsto (a+1)/2, \;\; a_1 \mapsto -A/2, \;\; n \mapsto 2n \tag{2.25}$$

we see that (2.23) coincides with (1.4).

Let us now consider the specialization (2.17) in (2.4). We must first give the form of (2.18) in the case that $n \mapsto n_2$, $\lambda \mapsto \kappa$, $\ell(\kappa) = n_1$ with $n_1 \le n_2$ so that $\kappa_{n_1+1} = \cdots = \kappa_{n_2} = 0$. Then with $n^* = n_1$ and $r_j := \kappa_j + n_1 - j$, manipulation of (2.18) shows

$$\begin{aligned} s_\kappa(1, t, \ldots, t^{n_2-1}) &= t^{-\sum_{j=1}^{n_2-n_1} j(j-1)} t^{-n_1 \sum_{j=1}^{n_2-n_1} j} \frac{t^{-\sum_{j=1}^{n_2}(j-1)(n_1-j)}}{\prod_{l=1}^{n_2-1}(t;t)_l} \\ &\times \prod_{i=1}^{n_2-n_1-1} (t;t)_i \prod_{i=1}^{n_1} \frac{(t;t)_{r_i+n_2-n_1}}{(t;t)_{r_i}} \prod_{1 \le i < j \le n_1} (t^{r_j} - t^{r_i}) \end{aligned} \tag{2.26}$$

where the first product in the second line must be replaced by unity if $n_2 = n_1, n_1+1$. Substituting this result, and (2.18) with $n \mapsto n_1$, $n^* = n_1$, $\lambda \mapsto \mu$ in (2.4) we deduce an interpretation of the PDF (1.2) in the context of a last passage percolation model.

Proposition 2. *Let $n_2 \ge n_1$. On each site of the $n_1 \times (n_2+1)$ square lattice specify a non-negative integer $x_{i,j}$ according to the probability distribution*

$$\begin{aligned} \Pr(x_{i,j} = k) &= (1 - z^2 t^{i+j-2})(z^2 t^{i+j-2})^k, \quad j \ne n_2 + 1 \\ \Pr(x_{i,n_2+1} = k) &= (1 - \alpha z t^{i-1})(\alpha z t^{i-1})^k. \end{aligned} \tag{2.27}$$

The joint probability that a configuration $[x_{i,j}]$ gives, under the RSK correspondence, a pair of tableaux of shape μ, one of content n_1 and the other of content n_2+1, and that the subconfiguration $[x_{i,j}]_{\substack{i=1,\ldots,n_1 \\ j=1,\ldots,n_2}}$ gives a pair of tableaux of shape κ, one of content n_1 and the other of content n_2, is non-zero if and only if

$$h_1 \ge r_1 > h_2 \ge r_2 > \cdots > h_{n_1} \ge r_{n_1} \ge 0, \tag{2.28}$$

where $h_j := \mu_j + n_1 - j$ and $r_j := \kappa_j + n_1 - j$. Furthermore the joint probability then has the explicit form

$$k_{n_1,n_2}(z,\alpha,t) z^{\sum_{j=1}^{n_1}(h_j+r_j)} \alpha^{\sum_{j=1}^{n_1}(h_j-r_j)} \prod_{i=1}^{n_1} \frac{(t;t)_{r_i+n_2-n_1}}{(t;t)_{r_i}} \prod_{1 \le i<j \le n_1} (t^{h_j} - t^{h_i})(t^{r_j} - t^{r_i}), \tag{2.29}$$

$$\begin{aligned} k_{n_1,n_2}(z,\alpha,t) &:= z^{-2\sum_{j=1}^{n_1}(n_1-j)} \frac{t^{-\sum_{j=1}^{n_1}(j-1)(n_1-j)}}{\prod_{l=1}^{n_1-1}(t;t)_l} \\ &\quad \times t^{-\sum_{j=1}^{n_2-n_1} j(j-1)} t^{-n_1\sum_{j=1}^{n_2-n_1} j} \frac{t^{-\sum_{j=1}^{n_2}(j-1)(n_1-j)}}{\prod_{l=1}^{n_2-1}(t;t)_l} \\ &\quad \times \prod_{l=1}^{n_2-n_1-1} (t;t)_l \prod_{i=1}^{n_1}\prod_{j=1}^{n_2} (1 - z^2 t^{i+j-2}) \prod_{i=1}^{n_1} (1 - \alpha z t^{i-1}). \end{aligned} \tag{2.30}$$

In the Jacobi limit (2.21) (with the additional scaled quantity $r_j/L =: y_j$), (2.29) multiplied by L^{2n_1} tends to the PDF

$$\begin{aligned} &\tilde{k}_{n_1,n_2}(a,a_1) \prod_{i=1}^{n_1} (1 - e^{-y_i})^{n_2-n_1} \prod_{1 \le i<j \le n_1} (e^{-y_j} - e^{-y_i})(e^{-x_j} - e^{-x_i}) \\ &\times e^{-a\sum_{j=1}^{n_1}(x_j+y_j)} e^{-a_1\sum_{j=1}^{n_1}(x_j-y_j)}, \end{aligned} \tag{2.31}$$

$$\tilde{k}_{n_1,n_2}(a,a_1) := \frac{\prod_{l=1}^{n_2-n_1-1} l!}{(\prod_{l=1}^{n_1-1} l!)(\prod_{l=1}^{n_2-1} l!)} \frac{\Gamma(a+a_1+n_1)}{\Gamma(a+a_1)} \prod_{i=1}^{n_1} \frac{\Gamma(2a+i-1+n_2)}{\Gamma(2a+i-1)}, \tag{2.32}$$

where it is required that

$$x_1 > y_1 > x_2 > y_2 > \cdots > x_{n_1} > y_{n_1} > 0. \tag{2.33}$$

Analogous to (2.25), after the change of variables and replacement of parameters

$$e^{-x_j} \mapsto x_{2n+1-2j}, \;\; e^{-y_j} \mapsto x_{2n+2-2j}, \;\; a \mapsto (a+1)/2, \;\; a_1 \mapsto -A/2 \tag{2.34}$$

we see that with $n_2 = n_1 = n$ (2.31) coincides with (1.5).

To specialize (2.4) modified by the replacement (2.6) according to (2.17), we note that for this to be non-zero (2.7) gives we require $\ell(\mu) \le n_2 + 1$. Making the replacements $\kappa \mapsto \mu$, $n_2 \mapsto n_1$, $n_1 \mapsto n_2 + 1$, $r_j \mapsto h_j := \kappa_j + n_2 + 1 - j$ in (2.26) shows

$$\begin{aligned} s_\mu(1,t,\dots,t^{n_1-1}) &= t^{-\sum_{j=1}^{n_1-n_2-1} j(j-1)} t^{-(n_2+1)\sum_{j=1}^{n_2-(n_1+1)}(j-1)} \frac{t^{-\sum_{j=1}^{n_1}(j-1)(n_2+1-j)}}{\prod_{l=1}^{n_1-1}(t;t)_l} \\ &\times \prod_{l=1}^{n_1-n_2-2}(t;t)_l \prod_{i=1}^{n_2+1} \frac{(t;t)_{h_i+n_1-(n_2+1)}}{(t;t)_{h_i}} \prod_{1\le i<j\le n_2+1}(t^{h_j}-t^{h_i}). \end{aligned} \tag{2.35}$$

Substituting this result, and (2.18) with $n \mapsto n_2$, $n^* = n_2$, $\lambda \mapsto \kappa$, $h_j \mapsto r_j := \kappa_j + n_2 - j$ we deduce the analogue of Proposition 2 in the case $n_1 > n_2$.

Proposition 3. *Let $n_1 > n_2$. On each site of the $n_1 \times (n_2+1)$ square lattice specify a non-negative integer $x_{i,j}$ according to the probability distribution (2.27). The joint probability that a configuration $[x_{i,j}]$ gives, under the RSK correspondence, a pair of tableaux of shape μ, one of content n_1 and the other of content n_2+1, and that the subconfiguration $[x_{i,j}]_{\substack{i=1,\dots,n_1 \\ j=1,\dots,n_2}}$ gives a pair of tableaux of shape κ, one of content n_1 and the other of content n_2, is non-zero if and only if*

$$h_1 \ge r_1 > h_2 \ge r_2 > \cdots > h_{n_2} \ge r_{n_2} > h_{n_2+1} \ge 0, \tag{2.36}$$

where $h_j := \mu_j + n_2 + 1 - j$ and $r_j := \kappa_j + n_2 + 1 - j$. Furthermore the joint probability has the explicit form

$$\begin{aligned} K_{n_1,n_2}(\alpha,z,t) z^{\sum_{j=1}^{n_2}(h_j+r_j)} \alpha^{\sum_{j=1}^{n_2}(h_j-r_j)} (z\alpha)^{h_{n_2+1}} \prod_{i=1}^{n_2+1} \frac{(t;t)_{h_i+n_1-(n_2+1)}}{(t;t)_{h_i}} \prod_{1\le i<j\le n_2+1}(t^{h_j}-t^{h_i}) \\ \times \prod_{1\le i<j\le n_2}(t^{r_j}-t^{r_i}), \end{aligned} \tag{2.37}$$

$$\begin{aligned} K_{n_1,n_2}(\alpha,z,t) \;\; &:= \;\; z^{-2\sum_{j=1}^{n_2}(n_2-j)} \frac{t^{-\sum_{j=1}^{n_2}(j-1)(n_2-j)}}{\prod_{l=1}^{n_2-1}(t;t)_l} \\ &\times t^{-\sum_{j=1}^{n_1-n_2-1} j(j-1)} t^{-(n_2+1)\sum_{j=1}^{n_1-n_2-1} j} \frac{t^{-\sum_{j=1}^{n_1}(j-1)(n_2+1-j)}}{\prod_{l=1}^{n_1-1}(t;t)_l} \\ &\times \prod_{l=1}^{n_1-n_2-2}(t;t)_l \prod_{i=1}^{n_1}\prod_{j=1}^{n_2}(1-z^2t^{i+j-2}) \prod_{i=1}^{n_1}(1-\alpha z t^{i-1}). \end{aligned} \tag{2.38}$$

In the Jacobi limit, (2.37) multiplied by L^{2n_2+1} tends to the PDF

$$\tilde{K}_{n_1,n_2}(a,a_1)\prod_{i=1}^{n_2+1}(1-e^{-x_i})^{n_1-(n_2+1)}\prod_{1\le i<j\le n_2+1}(e^{-x_j}-e^{-x_i})\prod_{1\le i<j\le n_2}(e^{-y_j}-e^{-y_i})$$
$$\times e^{-a\sum_{j=1}^{n_2}(x_j+y_j)}e^{-a_1\sum_{j=1}^{n_2}(x_j-y_j)}e^{-(a+a_1)x_{n_2+1}}, \tag{2.39}$$

$$\tilde{K}_{n_1,n_2}(a,a_1):=\frac{\prod_{l=1}^{n_1-n_2-2}l!}{(\prod_{l=1}^{n_1-1}l!)(\prod_{l=1}^{n_2-1}l!)}\frac{\Gamma(a+a_1+n_1)}{\Gamma(a+a_1)}\prod_{i=1}^{n_1}\frac{\Gamma(2a+i+n_2-1)}{\Gamma(2a+i-1)}, \tag{2.40}$$

where it is required that

$$x_1>y_1>x_2>y_2>\cdots>x_{n_2}>y_{n_2}>x_{n_2+1}>0. \tag{2.41}$$

With $n_1=n_2+1$, making the change of variables and replacements (2.34) in (2.39) gives the natural generalization of (1.4) to the case of an odd number of coordinates.

In the case $n_1=n_2=:n$, there is yet another combinatorial interpretation of the joint probability (2.29), which relates to a particular model (model (v)) introduced in [5]. Thus consider the $2n\times 2n$ square lattice of sites (i,j), $1\le i,j\le 2n$. For the triangular shaped region specified by $1\le i\le 2n-1$, $i\ge j$, $i\le 2n+1-j$ associate with each lattice site a non-negative integer $x_{i,j}$ chosen according to the probability distributions

$$\begin{aligned}
\Pr(x_{i,j}=k) &= \Pr(x_{i,2n+1-j}=k) = (1-z^2t^{i+j-2})(z^2t^{i+j-2})^k, \qquad i,j\le n\ (i\ne j)\\
\Pr(x_{i,i}=k) &= (1-\alpha z t^{i-1})(\alpha z t^{i-1})^k,\\
\Pr(x_{i,2n+1-i=k}) &= 0.
\end{aligned}$$

With the $x_{i,j}$ in this region thus chosen, specify $x_{i,j}$ at the remaining lattice sites in the square by the symmetry requirements

$$x_{j,i}=x_{i,j}, \qquad x_{2n+1-i,2n+1-j}=x_{i,j}. \tag{2.42}$$

The first of the symmetries in (2.42) implies that under the RSK correspondence the integer matrix maps to a single semi-standard tableau μ (of content $2n$), while the second symmetry implies that each row is of even length. At a more sophisticated level, the resulting tableau is constrained to be self-dual (invariant under Schützenberger involution). Although we don't present the details, using ideas from [4], from this one can show that with $h_j=\mu_{2j-1}/2+n-j$ and $r_j=\mu_{2j}/2+n-j$ the probability that $[x_{i,j}]$ maps to the semi-standard tableau μ is given by (2.29) with $n_1=n_2=n$.

We remark at this point that the special case $a=1$, $A=0$ of the Jacobi parameter dependent PDF (1.5) occurs as various probabilities in the work of Ciucu [11], and Krattenthaler [27] on perfect matchings (tilings) on the Aztec lattice with removed sites.

2.2 Laguerre limit

It has already been remarked that after writing $x_j\mapsto\frac{1}{2}(x_j+1)$, $(j=1,\dots,2n)$, then scaling the variables and parameters according to (1.6) and taking the limit $L\to\infty$, the parameter dependent Jacobi ensembles (1.4) and (1.5) reduce to the parameter dependent Laguerre ensembles (1.1) and (1.2) respectively. As first noticed by Johansson [23], Laguerre ensembles can be obtained directly from the Schur measure (2.3) by first setting all the variables equal (and thus choosing $t=1$ in (2.17)), then scaling the remaining parameters and variables as in (2.21). We thus obtain the following interpretation of the parameter dependent Laguerre ensembles.

Proposition 4. *First, on each site (i,j) of the $n \times n$ square lattice, specify a continuous exponential random variable*

$$\begin{aligned} \Pr(x_{i,j} \in [y, y+dy]) &= 2ae^{-2ay}\, dy, \qquad i < j \\ \Pr(x_{i,i} \in [y, y+dy]) &= (a+a_1)e^{-(a+a_1)y}\, dy \end{aligned}$$

and impose the symmetry constraint that $x_{i,j} = x_{j,i}$ for $i > j$. Then the probability density that a configuration $[x_{i,j}]$ gives, under the continuous RSK correspondence of Appendix A, a non-intersecting path configuration with maximum displacement x_l at level-l, is given by (see also [3])

$$\frac{(a+a_1)^n (2a)^{n(n-1)/2}}{\prod_{l=1}^{n-1} l!} e^{-a\sum_{j=1}^n x_j} e^{-a_1 \sum_{j=1}^n (-1)^{j-1} x_j} \prod_{1 \le i < j \le n} (x_i - x_j) \tag{2.43}$$

where $x_1 > x_2 > \cdots > x_n > 0$. For $n \mapsto 2n$ this is equivalent to (1.1).

Second, on each site (i,j) of the $n_1 \times (n_2+1)$ square lattice specify a continuous exponential random variable

$$\begin{aligned} \Pr(x_{ij} \in [y, y+dy]) &= 2ae^{-2ay}\, dy, \qquad j \ne n_2+1 \\ \Pr(x_{i\, n_2+1} \in [y, y+dy]) &= (a+a_1)e^{-(a+a_1)y}\, dy. \end{aligned} \tag{2.44}$$

Then for $n_2 \ge n_1$ the joint probability density that a configuration $[x_{i,j}]$ gives, under the continuous RSK correspondence of Appendix A, a non-intersecting path configuration with maximum displacement x_l at level-l, and that the subconfiguration $[x_{i,j}]_{\substack{i=1,\dots,n_1 \\ j=1,\dots,n_2}}$ gives a non-intersecting path configuration with maximum displacement y_l at level-l is non-zero if and only if the interlacing condition (2.33) holds, when it has the explicit form

$$\begin{aligned} &\frac{(2a)^{n_1 n_2}(a+a_1)^{n_1} \prod_{l=1}^{n_2-n_1-1} l!}{\prod_{l=1}^{n_1-1} l! \prod_{l=1}^{n_2-1} l!} e^{-a\sum_{j=1}^{n_1}(x_j+y_j)} e^{-a_1\sum_{j=1}^{n_1}(x_j-y_j)} \prod_{i=1}^{n_1} y_i^{n_2-n_1} \\ &\times \prod_{1 \le i < j \le n_1} (x_i - x_j)(y_i - y_j) \end{aligned} \tag{2.45}$$

In the case $n_1 = n_2$ this is equivalent to the PDF (1.2). For $n_2 < n_1$, the same joint probability density is non-zero if and only if the interlacing condition (2.41) holds, when it has the explicit form

$$\begin{aligned} &\frac{(2a)^{n_1 n_2}(a+a_1)^{n_1} \prod_{l=1}^{n_1-n_2-2} l!}{\prod_{l=1}^{n_1-1} l! \prod_{l=1}^{n_2-1} l!} e^{-a\sum_{j=1}^{n_2}(x_j+y_j)} e^{-a_1\sum_{j=1}^{n_2}(x_j-y_j)} \prod_{i=1}^{n_2+1} y_i^{n_2-n_1} \\ &\times \prod_{1 \le i < j \le n_2+1} (x_i - x_j) \prod_{1 \le i < j \le n_2} (y_i - y_j) \end{aligned} \tag{2.46}$$

2.3 Gaussian limit

It was pointed out by Baryshnikov [6], upon interpreting a result of Glynn and Whitt [21], that for $x_{i,j}$ i.i.d. random variables with finite variance, the quantity $L(n_1, n_2)$ specified by (2.2) has a universal scaled form in the limit $n_1 \to \infty$, independent of the details of the distribution. This universal form is the PDF for the distribution of the largest eigenvalue in the GUE of $n_2 \times n_2$ random complex Hermitian matrices, which have the joint eigenvalue probability density

$$\frac{1}{C} \prod_{l=1}^{n_2} e^{-x_l^2} \prod_{1 \le j < k \le n_2} (x_k - x_j)^2. \tag{2.47}$$

It follows that with $a = a_1$ in (2.44) so as to obtain i.i.d. random variables, we can expect to obtain a Gaussian type ensemble by taking the scaled $n_1 \to \infty$ limit in the joint probability (2.46). To see that this occurs requires nothing more than the classical transition between the Laguerre and Gaussian weights,

$$\lim_{c\to\infty} e^c e^{-cx} x^c \Big|_{x\mapsto 1+x\sqrt{2/c}} = e^{-x^2},$$

allowing us to derive the following result.

Proposition 5. *In (2.46) taking the Gaussian limit by setting*

$$a = a_1,\ \ 2a = n_1 - (n_2+1) =: c,\ \ x_i \mapsto 1 + x_i\sqrt{2/c},\ \ y_i \mapsto 1 + y_i\sqrt{2/c},\ \ n_1 \to \infty,$$

gives the PDF

$$\frac{2^{n_2(n_2+1)/2}}{\pi^{(n_2+1)/2}} \prod_{i=1}^{n_2+1} e^{-x_i^2} \prod_{1\le i<j\le n_2+1} (x_i - x_j) \prod_{1\le i<j\le n_2} (y_i - y_j) \tag{2.48}$$

where it is required that

$$\infty > x_1 > y_1 > x_2 > y_2 > \cdots > x_{n_2} > y_{n_2} > x_{n_2+1} > -\infty.$$

2.4 Limit to a biorthogonal Jacobi ensemble

In the joint probability (2.4) let us generalize the specialization (2.17) so that it involves two distinct sets of t variables and two distinct z variables, and thus choose

$$(a_1,\dots,a_{n_1}) = (z_1, z_1t_1, z_1t_1^2,\dots,z_1t_1^{n_1-1}), \quad (b_1,\dots,b_{n_2}) = (z_2, z_2t_2, z_2t_2^2,\dots,z_2t_2^{n_2-1}). \tag{2.49}$$

Using the Schur function evaluation formulas (2.18) and (2.26) we can readily write down the generalization of (2.29) and (2.37). Furthermore, the Jacobi limit of these generalizations can be computed. Let us make note of the explicit form in the case of (2.29).

Proposition 6. *Consider the generalization of (2.29) obtained by specializing (2.4) by (2.49). Let*

$$z_1 = e^{-a/L}, z_2 = e^{-\bar a/L}, t_1 = e^{-1/L}, t_2 = e^{-c/L}, \alpha = e^{-a_1/L}, h_j/L = x_j, r_j/L = y_j.$$

Then as $L \to \infty$, this probability multiplied by L^{2n_1} tends to the PDF

$$\begin{aligned}\tilde k^*_{n_1,n_2}(a,\bar a,a_1,c) &\prod_{i=1}^{n_1} (1-e^{-cy_i})^{n_2-n_1} \prod_{1\le i<j\le n_1} (e^{-cy_j} - e^{-cy_i})(e^{-x_j} - e^{-x_i}) \\ &\times e^{-a\sum_{j=1}^{n_1} x_j} e^{-\bar a \sum_{j=1}^{n_1} y_j} e^{-a_1\sum_{j=1}^{n_1}(x_j - y_j)},\end{aligned} \tag{2.50}$$

$$\tilde k^*_{n_1,n_2}(a,\bar a,a_1,c) := \frac{\prod_{l=1}^{n_2-n_1-1} c^l l!}{(\prod_{l=1}^{n_1-1} l!)(\prod_{l=1}^{n_2-1} c^l l!)} \frac{\Gamma(a+a_1+n_1)}{\Gamma(a+a_1)} \prod_{j=1}^{n_2} \frac{\Gamma(a+\bar a + c(j-1) + n_2)}{\Gamma(a+\bar a + j - 1)}. \tag{2.51}$$

The Jacobi limit of the Schur function identity (2.15) tells us that if we integrate (2.50) over $x_1,\dots,x_{n_1}$ we obtain the $a_1 \to \infty$ scaled limit (scaled by $a_1^{-n_1}$) of the same PDF, and thus

$$\begin{aligned}\Big(\lim_{a_1\to\infty} a_1^{-n_1} k^*_{n_1,n_2}(a,\bar a,a_1,c)\Big) &e^{-(a+\bar a)\sum_{j=1}^{n_1} y_j} \prod_{i=1}^{n_1} (1-e^{-cy_i})^{n_2-n_1} \\ &\times \prod_{1\le i<j\le n_1} (e^{-cy_j} - e^{-cy_i})(e^{-y_j} - e^{-y_i}).\end{aligned} \tag{2.52}$$

In the case $n_1 = n_2 = n$ this same PDF was derived in (A.10) of Appendix A from the continuous version of the RSK correspondence, giving the PDF for the event that the $n_1 \times n_2$ lattice of non-negative random variables $[x_{i,j}]$ distributed according to

$$\Pr(x_{ij} = y) = (i - 1 + c(j-1) + a + \bar{a}) e^{-y(i-1+c(j-1)+a+\bar{a})},$$

gives a polynuclear growth model with height of the level-l path y_l. After the change of variables and replacement of parameters $e^{-y_j} \mapsto y_{n+1-j}$, $(a + \bar{a}) \mapsto (\alpha + 1)$ we obtain the PDF

$$\Big(\lim_{a_1 \to \infty} a_1^{-n_1} k^*_{n_1,n_2}(a, \bar{a}, a_1, c) \Big) \prod_{i=1}^{n_1} y_i^\alpha (1 - y_i^c)^{n_2 - n_1} \prod_{1 \le i < j \le n_1} (y_j^c - y_i^c)(y_j - y_i) \tag{2.53}$$

where $1 > y_1 > \cdots > y_n > 0$. For general $\alpha > -1$, $c > 0$ and with $n_1 = n_2$ the k-point distribution corresponding to (2.53) has been computed by Borodin [10]. As the method required to accomplish this task made use of ideas from the theory of biorthogonal systems, it was referred to as the biorthogonal Jacobi ensemble. The PDF (2.50) represents a more general biorthogonal Jacobi ensemble, but the correlations for both sets of variables are yet to be computed.

We remark that from (2.53) we can take a Laguerre and a Gaussian limit. The correlations for both cases were also computed in [10].

2.5 Distribution functions

Let us denote by $E(j; I; \mathrm{PDF})$ the probability that the interval I of the specified PDF (for this we will use the corresponding equation number) contains exactly j eigenvalues (for the sake of definiteness in terminology we will regard the PDFs as measures for eigenvalues). Similarly, let us denote by $E^{(\cdot)}(j; I; \mathrm{PDF})$ the same quantity except that only $(\cdot)$ =(e)ven labelled or $(\cdot)$ =(o)dd labelled eigenvalues are being observed. Let us suppose now that $I = (s, \infty)$ where s is inside the support of the PDF. Then as discussed in [19], knowledge of $\{E(j; I; \mathrm{PDF})\}$ is equivalent to knowledge of $\{E^{(\cdot)}(j; I; \mathrm{PDF})\}$. Furthermore $p(k-1; s; \mathrm{ME})$ — the distribution function of the kth eigenvalue from the right — is determined by $\{E(j; I; \mathrm{PDF})\}$. The quantities $E(j; I; \mathrm{PDF})$, $E^{(\cdot)}(j; I; \mathrm{PDF})$, $p(k-1; s; \mathrm{ME})$ for the PDFs (1.1), (1.2), (1.4), (1.5) are discussed in [19], as are the scaled limits of these quantities. Here we want to use knowledge of the so called hard edge scalings from [19] to identify scales associated to the large eigenvalues of the particular continuous RSK measures (2.23) and (2.31).

In (1.4), (1.5) we first change variables $x_j \mapsto (1 - x_{2n+1-j})/2$ for consistency with [19]. We know from [19] that then, with $2n = N$ in (1.4) and $n = N$ in (1.5), the large eigenvalues have a well defined scaled limit obtained by setting

$$x_j = 1 - \frac{X_j}{2N^2}, \qquad A = 4N^2 \bar{\alpha}$$

(we use $\bar{\alpha}$ rather than α as used in [19] to avoid confusion with α as used in (2.12)) and taking $N \to \infty$. In particular

$$\lim_{N\to\infty} E^{(\mathrm{e})}\Big(p; (1 - \frac{s^2}{2N^2}, 1); (1.4)\Big|_{A=4N^2\bar{\alpha}}\Big) = E(p; (0,s); \mathrm{SE}^{\mathrm{hard}, a+1}) \tag{2.54}$$

$$\lim_{N\to\infty} E^{(\mathrm{e})}\Big(p; (1 - \frac{s^2}{2N^2}, 1); (1.5)\Big|_{A=4N^2\bar{\alpha}}\Big) = E(p; (0,s); \mathrm{UE}^{\mathrm{hard}, a}) \tag{2.55}$$

$$\lim_{N\to\infty} E^{(\mathrm{o})}\Big(p; (1 - \frac{s^2}{2N^2}, 1); (1.4)\Big|_{A=4N^2\bar{\alpha}}\Big) = E^{(\mathrm{o})}(p; (0,s); \mathrm{OE}^{\bar{\alpha}, a}) \tag{2.56}$$

$$\lim_{N\to\infty} E^{(\mathrm{e})}\Big(p; (1 - \frac{s^2}{2N^2}, 1); (1.5)\Big|_{A=4N^2\bar{\alpha}}\Big) = E(p; (0,s); (\mathrm{OE} \cup \mathrm{OE})^{\bar{\alpha}, a}) \tag{2.57}$$

Notice that (2.54), (2.55) are independent of the parameter $\bar{\alpha}$. We have already commented that the PDFs (2.23) and (2.31) (the latter with $n_1 = n_2 = n$) are related to (1.4) and (1.5) by a change of variables. Hence it follows that

$$\begin{aligned}
\lim_{N\to\infty} E^{(\cdot)}\Big(p; (2\log N - \log\frac{s}{4}, \infty); (2.23)\Big|_{\substack{n=N, a\mapsto(a+1)/2 \\ a_1=-2N^2\bar{\alpha}}}\Big) &= E^{(\cdot)}(p; (0,s); (\mathrm{OE})^{\bar{\alpha},a}) \\
\lim_{N\to\infty} E^{(\cdot)}\Big(p; (2\log N - \log\frac{s}{4}, \infty); (2.31)\Big|_{\substack{n_1=n_2=N, a\mapsto(a+1)/2 \\ a_1=-2N^2\bar{\alpha}}}\Big) &= E^{(\cdot)}(p; (0,s); (\mathrm{OE}\cup\mathrm{OE})^{\bar{\alpha},a}) \qquad (2.58)
\end{aligned}$$

Also of interest is the scaled form of the gap probability for the continuous RSK measure (2.52). Now results in [10] imply

$$\lim_{n_1\to\infty} E\Big(p; (0, \frac{s}{4N^{1+1/c}}; (2.53)\Big|_{n_2=n_1}\Big) = \frac{(-1)^p}{p!}\frac{\partial^p}{\partial\xi^p}\det(1-\xi K^{(\alpha,c)})\Big|_{\xi=1} \qquad (2.59)$$

where $K^{(\alpha,c)}$ is the integral operator supported on $(0,s)$ with the kernel

$$\begin{aligned}
K^{(\alpha,c)}(x,y) &= \frac{c}{4}\int_0^1 J_{(\alpha+1)/c,1/c}(xt/4)J_{\alpha+1,c}((yt/4)^c)t^\alpha\,dt, \\
J_{a,b}(x) &:= \sum_{m=0}^\infty \frac{(-x)^m}{m!\Gamma(a+bm)}.
\end{aligned}$$

It follows from the relationship between (2.53) and (2.52) that

$$\lim_{n_1\to\infty} E\Big(p; ((1+1/c)\log N - \log\frac{s}{4}, \infty); (2.52)\Big|_{\substack{n_2=n_1 \\ a+\bar{a}=\alpha+1}}\Big) = \frac{(-1)^p}{p!}\frac{\partial^p}{\partial\xi^p}\det(1-\xi K^{(\alpha,c)})\Big|_{\xi=1}. \qquad (2.60)$$

Notice that in the case $c=1$ the scaled interval is the same as that in (2.58).

3 Interpolating ensembles from Macdonald polynomial theory

The recurrence (2.14) satisfied by the Schur polynomials is a special case of a more general recurrence satisfied by the Macdonald polynomials, as is the marginal probability (2.3) and the evaluation formula (2.18). This then allows a generalization of the joint probability (2.4) to the Macdonald setting. We will see that the probability (2.19), which in the last passage percolation problem results from imposing the symmetry constraint $x_{i,j} = x_{j,i}$ on the waiting times, is also a special case of the generalized joint probability.

The recurrence (2.14) can be used to define the Schur polynomials. Likewise we can define the (monic) Macdonald polynomials $P_\kappa(b_1,\dots,b_{n_2};q,t)$ by the recurrence [28, pg. 348]

$$\sum_{\kappa:\kappa\in R} \psi_{\mu/\kappa}(q,t)P_\kappa(b_1,\dots,b_{n_2};q,t)b_{n_2+1}^{|\mu|-|\kappa|} = P_\mu(b_1,\dots,b_{n_2+1};q,t) \qquad (3.1)$$

where with $f(x) := (tx;q)_\infty/(qx;q)_\infty$

$$\psi_{\mu/\kappa}(q,t) := \prod_{1\le i\le j\le \ell(\kappa)} \frac{f(q^{\kappa_i-\kappa_j}t^{j-i})f(q^{\mu_i-\mu_{j+1}}t^{j-i})}{f(q^{\mu_i-\kappa_j}t^{j-i})f(q^{\kappa_i-\mu_{j+1}}t^{j-i})}. \qquad (3.2)$$

The Schur polynomials are the special case $q=t$ of the Macdonald polynomials (note that with $q=t$, $f(x)=1$ and so $\psi_{\mu/\kappa}(q,q)=1$).

In the Macdonald theory, the generalization of the marginal probability (2.3) is

$$\prod_{i=1}^{n_1}\prod_{j=1}^{n_2}\frac{(a_ib_j;q)_\infty}{(ta_ib_j;q)_\infty}Q_\mu(a_1,\dots,a_{n_1};q,t)P_\mu(b_1,\dots,b_{n_2};q,t)$$

where

$$Q_\mu(a_1,\dots,a_{n_1};q,t) = \langle P_\mu, P_\mu\rangle^{-1} P_\mu(a_1,\dots,a_{n_1};q,t)$$

and $\langle\cdot,\cdot\rangle$ is a particular power sum inner product [28, pg. 309]. Thus the natural generalization of (2.4) is

$$\prod_{i=1}^{n_1}\prod_{j=1}^{n_2+1} \frac{(a_ib_j;q)_\infty}{(ta_ib_j;q)_\infty} Q_\mu(a_1,\dots,a_{n_1};q,t)P_\kappa(b_1,\dots,b_{n_2};q,t)\psi_{\mu/\kappa}(q,t)b_{n_2+1}^{|\mu|-|\kappa|} \tag{3.3}$$

where the parts of μ and κ are restricted by (2.5) in the case $n_2 \ge n_1$, and (2.7) in the case $n_1 > n_2$. Because of the identity (3.1), and the further Macdonald polynomial identity [28, special case of 6(a) pg. 352]

$$\prod_{i=1}^{n_1} \frac{(a_ib_{n_2+1};q)_\infty}{(ta_ib_{n_2+1};q)_\infty} \sum_{\mu:\mu\in R} Q_\mu(a_1,\dots,a_{n_1})\psi_{\mu/\kappa}(q,t)b_{n_2+1}^{|\mu|-|\kappa|} = Q_\kappa(a_1,\dots,a_{n_1}) \tag{3.4}$$

which generalizes (2.15), this furthermore has the interpretation as a joint probability density function on tableaux of shape μ and shape κ when μ/κ is a horizontal strip.

The Macdonald polynomials exhibit a generalization of the evaluation formula (2.18). Let $p_r := \sum_{j=1}^n x_j^r$ denote the power sum of degree r. Define a homomorphism

$$\varepsilon_{u,t}(p_r) = \frac{1-u^r}{1-t^r} \tag{3.5}$$

Then for any symmetric function f analytic in $x_1,\dots,x_n$ it is easy to see that

$$\varepsilon_{t^n,t}(f) = f(1,t,\dots,t^{n-1}), \tag{3.6}$$

so the evaluation of $\varepsilon_{u,t}(P_\mu)$ includes the generalization of (2.18) for the Macdonald polynomials. From [28, pg. 343] we have with $t=q^k$ and $\ell(\lambda)\le n$

$$\begin{aligned}
\varepsilon_{u,t}(P_\lambda) &= t^{\sum_{i=1}^n (i-1)\lambda_i} \prod_{1\le i<j\le n} (q^{\lambda_i-\lambda_j}t^{j-i};q)_k \prod_{i=1}^n \frac{(ut^{1-i};q)_{\lambda_i}}{(t;q)_{\lambda_i+k(n-i)}} && (3.7)\\
\varepsilon_{u,t}(Q_\lambda) &= t^{\sum_{i=1}^n (i-1)\lambda_i} \prod_{1\le i<j\le n} (q^{\lambda_i-\lambda_j+1}t^{j-i-1};q)_k \prod_{i=1}^n \frac{(ut^{1-i};q)_{\lambda_i}}{(q;q)_{\lambda_i+k(n-i)}} && (3.8)
\end{aligned}$$

Note that for n fixed these quantities are strictly positive for u small enough, but may become zero or negative if n is unrestricted. In fact a special choice of u in each case shows that the action of $\varepsilon_{u,t}$ may annihilate P_λ or Q_λ for all λ of length greater than a fixed size. Thus we see that

$$\varepsilon_{t^n,t}(P_\lambda)=0, \quad \varepsilon_{qt^{n-1},t}(Q_\lambda)=0, \qquad \text{for} \quad \ell(\lambda)>n, \tag{3.9}$$

while these same quantities are strictly positive for $\ell(\lambda)\le n$ (in the first case corresponding to the evaluation (3.6)).

In (3.3) we write $b_{n_2+1} =: \alpha$, replace a_i by z_1a_i $(i=1,\dots,n_1)$, b_j by z_2b_j $(j=1,\dots,n_2)$ and take the limit $n_1,n_2\to\infty$. We would now like to apply the homomorphism $\varepsilon_{u,t}$ to the functions of $\{a_i\}$ and the homomorphism $\varepsilon_{w,t}$ to the functions of $\{b_j\}$. For the Macdonald polynomials, after factoring out the z-dependence using homogeneity, this is done using the formulas (3.7) and (3.8). For the infinite products we recall the formula [28, pg. 310]

$$\prod_{i,j=1}^\infty \frac{(z_1z_2a_ib_j;q)_\infty}{(tz_1z_2a_ib_j;q)_\infty} = \prod_{n=1}^\infty \exp\Big(-\frac{(z_1z_2)^n}{n}\frac{1-t^n}{1-q^n}p_n(a)p_n(b)\Big),$$

from which the action (3.5) immediately implies

$$\begin{aligned}
\varepsilon_{u,t}^{\{a_i\}}\varepsilon_{w,t}^{\{b_i\}}\Big(\prod_{i,j=1}^{\infty}\frac{(z_1z_2a_ib_j;q)_\infty}{(tz_1z_2a_ib_j;q)_\infty}\Big) &= \prod_{n=1}^{\infty}\exp\Big(-\frac{(z_1z_2)^n}{n}\frac{(1-u^n)(1-w^n)}{(1-q^n)(1-t^n)}\Big)\\
&= \prod_{p=0}^{\infty}\frac{(z_1z_2t^p;q)_\infty(uwz_1z_2t^p;q)_\infty}{(uz_1z_2t^p;q)_\infty(wz_1z_2t^p;q)_\infty}\\
\varepsilon_{u,t}^{\{a_i\}}\prod_{i=1}^{\infty}\frac{(z_1a_ib_{n_2+1};q)_\infty}{(tz_1a_ib_{n_2+1};q)_\infty} &= \frac{(z_1b_{n_2+1};q)_\infty}{(uz_1b_{n_2+1};q)_\infty}.
\end{aligned}$$

Consequently the image of the joint probability (3.3) is given by

$$\frac{(z_1\alpha;q)_\infty}{(uz_1\alpha;q)_\infty}\prod_{p=0}^{\infty}\frac{(z_1z_2t^p;q)_\infty(uwz_1z_2t^p;q)_\infty}{(uz_1z_2t^p;q)_\infty(wz_1z_2t^p;q)_\infty}\varepsilon_{u,t}(Q_\mu)\varepsilon_{w,t}(P_\kappa)\psi_{\mu/\kappa}(q,t)z_1^{|\mu|}z_2^{|\kappa|}\alpha^{|\mu|-|\kappa|} \tag{3.10}$$

For general u and w (3.10) is not itself a meaningful joint probability on partitions μ,κ because when the number of parts becomes large enough it will become negative. However, according to (3.7) and (3.8) for the special choice $u=t^n$ or $w=t^{n-1}$ this does not happen but rather (3.10) vanishes when $\ell(\mu)>n$. Thus we are naturally led to two distinct joint probabilities on partitions μ,κ with μ/κ a horizontal strip,

$$\begin{aligned}
\mathrm{Pr}_{\mathrm{e}}(\mu,\kappa) &:= \frac{(z_1\alpha;q)_\infty}{(t^nz_1\alpha;q)_\infty}\prod_{p=0}^{\infty}\frac{(z_1z_2t^p;q)_\infty(wz_1z_2t^{p+n};q)_\infty}{(z_1z_2t^{p+n};q)_\infty(wz_1z_2t^p;q)_\infty}\\
&\quad\times\varepsilon_{t^n,t}(Q_\mu)\varepsilon_{w,t}(P_\kappa)\psi_{\mu/\kappa}(q,t)z_1^{|\mu|}z_2^{|\kappa|}\alpha^{|\mu|-|\kappa|}
\end{aligned}\tag{3.11}$$

for which

$$\mu_1\ge\kappa_1\ge\mu_2\ge\kappa_2\ge\cdots\ge\mu_n\ge\kappa_n\ge0, \tag{3.12}$$

and

$$\begin{aligned}
\mathrm{Pr}_{\mathrm{o}}(\mu,\kappa) &:= \frac{(z_1\alpha;q)_\infty}{(uz_1\alpha;q)_\infty}\prod_{p=0}^{\infty}\frac{(z_1z_2t^p;q)_\infty(uz_1z_2t^{p+n-1};q)_\infty}{(uz_1z_2t^p;q)_\infty(z_1z_2t^{p+n-1};q)_\infty}\\
&\quad\times\varepsilon_{u,t}(Q_\mu)\varepsilon_{t^{n-1},t}(P_\kappa)\psi_{\mu/\kappa}(q,t)z_1^{|\mu|}z_2^{|\kappa|}\alpha^{|\mu|-|\kappa|}
\end{aligned}\tag{3.13}$$

for which

$$\mu_1\ge\kappa_1\ge\mu_2\ge\kappa_2\ge\cdots\ge\mu_{n-1}\ge\kappa_{n-1}\ge\mu_n\ge0. \tag{3.14}$$

A short calculation shows that setting $w=q^{n_2-n_1+1}t^{n-1}, z_1=z_2=z, t=q$ in $\mathrm{Pr}_{\mathrm{e}}(\mu,\kappa)$ gives (2.29), while setting $u=q^{n_1-(n_2+1)}t^{n-1}, z_1=z_2=z, t=q$ in $\mathrm{Pr}_{\mathrm{o}}(\mu,\kappa)$ gives (2.37). Furthermore (3.11), (3.13) in the case $t=q^2$ reclaim (2.19). Thus straight forward simplification gives the following result.

Proposition 7. *Let $w=q^{-1}t^n$, $z_1=z$, $z_2=qz$, and $t=q^2$, and write $h_{2j-1}:=\mu_j+2n-(2j-1)$, $h_{2j}:=\kappa_j+2n-2j$. Then (3.11) reduces to (2.19) with $n\mapsto 2n$, $t\mapsto q$ in the latter. Similarly, let $u=qt^{n-1}$, $z_1=z$, $z_2=qz$ and $t=q^2$, and write $h_{2j-1}:=\mu_j+(2n-1)-(2j-1), h_{2j}:=\kappa_j+(2n-1)-2j$. Then (3.13) reduces to (2.19) with $n\mapsto 2n-1$, $t\mapsto q$ in the latter.*

The probabilities $\mathrm{Pr}_{\mathrm{e}}(\mu,\kappa)$ and $\mathrm{Pr}_{\mathrm{o}}(\mu,\kappa)$ exhibit a special property with respect to summation over μ. Thus it follows from (3.4) that

$$\begin{aligned}
\sum_\mu\mathrm{Pr}_{\mathrm{e}}(\mu,\kappa) &= \prod_{p=0}^{\infty}\frac{(z_1z_2t^p;q)_\infty(wz_1z_2t^{p+n};q)_\infty}{(z_1z_2t^{p+n};q)_\infty(wz_1z_2t^p;q)_\infty}\varepsilon_{t^n,t}(Q_\kappa)\varepsilon_{w,t}(P_\kappa)(z_1z_2)^{|\kappa|}\\
\sum_\mu\mathrm{Pr}_{\mathrm{o}}(\mu,\kappa) &= \prod_{p=0}^{\infty}\frac{(z_1z_2t^p;q)_\infty(uz_1z_2t^{p+n-1};q)_\infty}{(uz_1z_2t^p;q)_\infty(z_1z_2t^{p+n-1};q)_\infty}\varepsilon_{u,t}(Q_\kappa)\varepsilon_{t^{n-1},t}(P_\kappa)(z_1z_2)^{|\kappa|}
\end{aligned}\tag{3.15}$$

A special case of $\mathrm{Pr}_{\mathrm{o}}(\mu,\kappa)$ also exhibits a special property with respect to summation over κ. To see this we first note from (3.6) that

$$\varepsilon_{t^{n-1},t}(P_\kappa) = P_\kappa(1,t,\dots,t^{n-2})$$

and so for the κ dependent terms in (3.13) with $\alpha=1$, $z_2=t$ we have

$$\begin{aligned}\sum_\kappa \psi_{\mu/\kappa}(q,t)t^{|\kappa|}\varepsilon_{t^{n-1},t}(P_\kappa) &= \sum_\kappa \psi_{\mu/\kappa}(q,t)P_\kappa(t,t^2,\dots,t^{n-1}) \\ &= P_\mu(1,t,t^2,\dots,t^{n-1}) = \varepsilon_{t^n,t}(P_\mu)\end{aligned} \tag{3.16}$$

where the second equality follows from (3.1) and the fact that P_κ is a symmetric function. Thus

$$\sum_\kappa \mathrm{Pr}_{\mathrm{o}}(\mu,\kappa)\Big|_{\substack{\alpha=1\\ z_2=t}} = \frac{(z_1;q)_\infty}{(uz_1;q)_\infty}\prod_{p=0}^\infty \frac{(z_1t^{p+1};q)_\infty(uz_1t^{p+n};q)_\infty}{(uz_1t^{p+1};q)_\infty(z_1t^{p+n};q)_\infty}\varepsilon_{u,t}(Q_\mu)\varepsilon_{t^n,t}(P_\mu)z_1^{|\mu|}. \tag{3.17}$$

To use (3.16) in the case of $\mathrm{Pr}_{\mathrm{e}}(\mu,\kappa)$ we must set $w=t^{n-1}$. This in turn implies $\kappa_n=0$, so we see that no new identity results, but rather we reclaim the special case $u=t^n$ of (3.17). As made explicit in Appendix B, the identity (3.17) can be recognized as being equivalent to a special case of a q-integral due to Evans [13], and also as the $\nu=\emptyset$ case of Okounkov's q-integral representation of the Macdonald polynomial P_ν [31]. The structure afforded by (3.17) suggests a simplified derivation of the latter which is given in Appendix B. We remark too that (3.17) can be considered as a particular q,t generalization of a class of measures on partitions known as z-measures [9].

Because the probabilities (2.29), (2.37) and (2.19) can all be derived from (3.10), they all exhibit the special property (3.15). Of course in the cases of (2.29), (2.37) this identity is immediate from their interpretation as joint probabilities for tableaux of shape μ and tableaux of shape κ with μ/κ a horizontal strip. But the probability (2.19) has no such interpretation, and the identity implied by (3.15),

$$\begin{aligned}&\mathrm{even}\Big(c_n(\alpha,z,t)z^{\sum_{j=1}^n h_j}\alpha^{\sum_{j=1}^n(-1)^{j-1}h_j}\prod_{1\le i<j\le n}(t^{h_j}-t^{h_i})\Big) \\ &= \Big(\alpha^{[n/2]}c_n(\alpha,z,t)\Big)\Big|_{\alpha=0} z^{\sum_{j=1}^{[n/2]}h_{2j}}\prod_{1\le i<j\le n}(t^{h_j}-t^{h_i})\Big|_{\substack{h_{2j-1}=h_{2j}+1\ (j=1,\dots,[n/2])\\ h_n=0\ (n\ \mathrm{odd})}},\end{aligned} \tag{3.18}$$

(here the notation even( ) denotes the distribution of the even labelled coordinates $h_2,h_4,\dots,h_{2[n/2]}$), telling us that there is no dependence on α after summing out the odd labelled coordinates, cannot easily be anticipated.

3.1 $\mathrm{Pr}_{\mathrm{o}}(\mu,\kappa)$ and $\mathrm{Pr}_{\mathrm{e}}(\mu,\kappa)$ in the Jacobi limit

Consider (3.13) with $z_1=z$, $z_2=tq^{-1}\bar z$, $u=q^{\beta+1}t^{n-1}$, $t=q^k$. The Jacobi limit is obtained by setting

$$z=e^{-a/L}, \bar z=e^{-\bar a/L}, \alpha=e^{-a_1/L}, q=e^{-1/L}, \mu_j/L=x_j, \kappa_j/L=y_j, \tag{3.19}$$

multiplying (3.13) by L^{2n-1} and taking the limit $L\to\infty$. This gives the PDF

$$\begin{aligned}&C_n(a,\bar a,a_1,\beta,k)e^{-a\sum_{i=1}^n x_i}e^{-\bar a\sum_{i=1}^{n-1}y_i}e^{-a_1(\sum_{i=1}^n x_i-\sum_{j=1}^{n-1}y_j)}\prod_{i=1}^n(1-e^{-x_i})^\beta \\ &\times\prod_{1\le i<j\le n}|e^{-x_j}-e^{-x_i}|\prod_{1\le i<j\le n-1}|e^{-y_j}-e^{-y_i}|\prod_{i=1}^n\prod_{j=1}^{n-1}|e^{-x_j}-e^{-y_i}|^{k-1}, \\ &C_n(a,\bar a,a_1,\beta,k):=\frac{\Gamma(a+a_1+\beta+1+k(n-1))}{\Gamma(a+a_1)\Gamma(\beta+1)}\prod_{i=1}^{n-1}\frac{\Gamma(a+\bar a+\beta+k(n-1+i))}{\Gamma(ki)\Gamma(\beta+ki+1)\Gamma(a+\bar a-1+ik)}.\end{aligned}$$

Replacing $a, \bar{a}$ by $a+1, \bar{a}+1$ and changing variables $e^{-x_i} \mapsto x_{n+1-i}$, $e^{-y_i} \mapsto y_{n-i}$, this reads

$$
\begin{aligned}
&C_n(a+1, \bar{a}+1, a_1, \beta, k) \prod_{i=1}^{n} x_i^{a+a_1} (1-x_i)^{\beta} \prod_{j=1}^{n-1} y_j^{\bar{a}-a_1} \\
&\times \prod_{1 \le i < j \le n} |x_j - x_i| \prod_{1 \le i < j \le n-1} |y_j - y_i| \prod_{i=1}^{n} \prod_{j=1}^{n-1} |x_j - y_i|^{k-1}
\end{aligned}
\tag{3.20}
$$

and we require the analogue of the interlacing condition (3.14),

$$
1 > x_1 > y_1 > x_2 > y_2 > \cdots > y_{n-1} > x_n > 0. \tag{3.21}
$$

A natural generalization of (3.20) is to include a factor $\prod_{j=1}^{n-1}(1-y_j)^{\beta}$. To compute the corresponding normalization, we note that with R denoting the region (3.21), the Jacobi limit of the second identity in (3.15) tells us that

$$
\begin{aligned}
&\int_R dx_1 \cdots dx_n \prod_{i=1}^{n} x_i^{\alpha} (1-x_i)^{\beta} \prod_{1 \le i < j \le n} |x_i - x_j| \prod_{i=1}^{n} \prod_{j=1}^{n-1} |x_j - y_i|^{k-1} \\
&= \frac{\Gamma(1+\alpha)\Gamma(1+\beta)(\Gamma(k))^{n-1}}{\Gamma(2+\alpha+\beta+(n-1)k)} \prod_{i=1}^{n-1} y_i^{\alpha+k} (1-y_i)^{\beta+k} \prod_{1 \le i < j \le n-1} |y_i - y_j|^{2k-1}
\end{aligned}
\tag{3.22}
$$

This is an integration formula due to Anderson [2]. Furthermore, we have the well known Selberg integral evaluation

$$
\begin{aligned}
&\int_0^1 dt_1\, t_1^{\lambda_1} (1-t_1)^{\lambda_2} \cdots \int_0^1 dt_N\, t_N^{\lambda_1} (1-t_N)^{\lambda_2} \prod_{1 \le j < k \le N} |t_k - t_j|^{2\lambda} \\
&= \prod_{j=0}^{N-1} \frac{\Gamma(\lambda_1+1+j\lambda)\Gamma(\lambda_2+1+j\lambda)\Gamma(1+(j+1)\lambda)}{\Gamma(\lambda_1+\lambda_2+2+(N+j-1)\lambda)\Gamma(1+\lambda)} =: S_N(\lambda_1, \lambda_2, \lambda).
\end{aligned}
\tag{3.23}
$$

It follows from (3.22) and (3.23) that

$$
\begin{aligned}
J_{\rm o}^{(n,n-1)}(x,y) &:= \frac{\Gamma(2+\alpha+\beta+(n-1)k)}{\Gamma(1+\alpha)\Gamma(1+\beta)(\Gamma(k))^{n-1}} \frac{1}{S_{n-1}(\alpha+\alpha_1+k, \beta+\beta_1+k, k)} \prod_{i=1}^{n} x_i^{\alpha}(1-x_i)^{\beta} \\
&\times \prod_{1 \le i < j \le n} |x_j - x_i| \prod_{i=1}^{n-1} y_i^{\alpha_1} (1-y_i)^{\beta_1} \prod_{1 \le i < j \le n-1} |y_j - y_i| \prod_{i=1}^{n} \prod_{j=1}^{n-1} |x_j - y_i|^{k-1}
\end{aligned}
\tag{3.24}
$$

is a correctly normalized joint PDF. With $\alpha = a + a_1$, $\alpha_1 = \bar{a} - a_1$, $\beta_1 = 0$ it coincides with (3.20).

Key properties of (3.24) are

$$
\begin{aligned}
\int_R dx_1 \cdots dx_n\, J_{\rm o}^{(n,n-1)}(x,y) &= \frac{1}{S_{n-1}(\alpha+\alpha_1+k, \beta+\beta_1+k, k)} \\
&\times \prod_{i=1}^{n-1} y_i^{\alpha+\alpha_1+k} (1-y_i)^{\beta+\beta_1+k} \prod_{1 \le i < j \le n-1} |y_i - y_j|^{2k} =: J_{\rm o}^{(-,n-1)}(y),
\end{aligned}
\tag{3.25}
$$

which follows immediately from (3.22), and

$$
\begin{aligned}
&\int_R dy_1 \cdots dy_{n-1}\, J_{\rm o}^{(n,n-1)}(x,y) \Big|_{\alpha_1=\beta_1=0} \\
&= \frac{1}{S_n(\alpha, \beta, k)} \prod_{i=1}^{n} x_i^{\alpha} (1-x_i)^{\beta} \prod_{1 \le i < j \le n} |x_i - x_j|^{2k} =: J_{\rm o}^{(n,-)}(x).
\end{aligned}
\tag{3.26}
$$

which can be deduced from the Jacobi limit of (3.17). Like (3.22) (and thus (3.25)), (3.26) is an integration formula due to Anderson [2].

Consider now (3.11), and put $z_1 = z$, $z_2 = tq^{-1}z$, $w = t^{\beta_2+n}$, $t = q^k$. The Jacobi limit is obtained by scaling the parameters according to (3.19), multiplying (3.11) by L^{2n} and taking the limit $L \to \infty$. One thus obtains the PDF

$$\begin{aligned} &K_n(a,\bar a,a_1,\beta,k) e^{-a\sum_{i=1}^n x_i} e^{-\bar a\sum_{i=1}^n y_i} e^{-a_1\sum_{i=1}^n (x_i-y_i)} \prod_{i=1}^n (1-e^{-y_i})^{k\beta_2} \\ &\quad \times \prod_{1\le i<j\le n} |e^{-x_j}-e^{-x_i}||e^{-y_j}-e^{-y_i}| \prod_{i,j=1}^n |e^{-x_j}-e^{-y_i}|^{k-1}, \\ &K_n(a,\bar a,a_1,\beta,k) := \frac{\Gamma(a+a_1+kn))}{\Gamma(a+a_1)} \frac{1}{\prod_{i=1}^n \Gamma(k(\beta_2+i))} \prod_{j=1}^{n+\beta_2} \frac{\Gamma(a+\bar a-1+k(j+1+n))}{\Gamma(a+\bar a-1+kj)}. \end{aligned}$$

By changing variables $e^{-x_i} \mapsto x_{n+1-i}$, $e^{-y_i} \mapsto y_{n-i}$ and replacing $a, \bar a$ by $a+1, \bar a+1$ we obtain from this the PDF

$$K_n(a+1,\bar a+1,a_1,\beta,k) \prod_{i=1}^n x_i^{a+a_1} y_i^{\bar a-a_1} (1-y_i)^{k\beta_2} \prod_{1\le i<j\le n} |x_j-x_i||y_j-y_i| \prod_{i,j=1}^n |x_j-y_i|^{k-1} \tag{3.27}$$

and we require the analogue of the interlacing condition (3.14),

$$1 > y_1 > x_1 > y_2 > x_2 > \cdots > y_n > x_n > 0. \tag{3.28}$$

Unlike the situation with (3.20), it is not a natural generalization of (3.27) to include an extra factor (here $\prod_{i=1}^n (1-x_i)^\beta$). Only with this extra factor absent can we compute the integral of the x's according to the Jacobi limit of the first identity in (3.15). Before stating this result, let us first rename some of the parameters and manipulate the normalization so that (3.27) reads

$$\begin{aligned} J_{\mathrm e}^{(n,n)}(x,y) &:= \frac{\Gamma(1+\alpha+nk)}{\Gamma(1+\alpha)(\Gamma(k))^n} \frac{1}{S_n(\alpha+\alpha_1+k,\beta_1,k)} \prod_{i=1}^n x_i^\alpha y_i^{\alpha_1} (1-y_i)^{\beta_1} \\ &\quad \times \prod_{1\le i<j\le n} |x_j-x_i||y_j-y_i| \prod_{i,j=1}^n |x_j-y_i|^{k-1} \end{aligned} \tag{3.29}$$

In terms of this quantity the Jacobi limit of the first identity in (3.15) reads

$$\begin{aligned} \int_{\tilde R} dx_1\cdots dx_n\, J_{\mathrm e}^{(n,n)}(x,y) &= \frac{1}{S_n(\alpha+\alpha_1+k,\beta_1,k)} \\ &\quad \times \prod_{i=1}^n y_i^{\alpha+\alpha_1+k}(1-y_i)^{\beta_1} \prod_{1\le i<j\le n} |y_i-y_j|^{2k} =: J_{\mathrm e}^{(-,n)}(y), \end{aligned} \tag{3.30}$$

where $\tilde R$ denotes the region (3.28). It follows from this that we also have

$$\begin{aligned} \int_{\tilde R} dy_1\cdots dy_n\, J_{\mathrm e}^{(n,n)}(x,y)\Big|_{\alpha_1=0} &= \frac{1}{S_n(\alpha,\beta_1+k,k)} \\ &\quad \times \prod_{i=1}^n x_i^{\alpha}(1-x_i)^{\beta_1+k} \prod_{1\le i<j\le n} |x_i-x_j|^{2k} =: J_{\mathrm e}^{(n,-)}(x), \end{aligned} \tag{3.31}$$

4 Random matrix interpretation of the Anderson density

With $J_{\rm o}^{(n,n-1)}(x,y)$, $J_{\rm o}^{(n,\text{-})}(x)$ and $J_{\rm o}^{(\text{-},n-1)}(y)$ defined by (3.24), (3.26) and (3.25) respectively, we can construct the conditional PDF's

$$\frac{J_{\rm o}^{(n,n-1)}(x,y)|_{\alpha_1=\beta_1=0}}{J_{\rm o}^{(n,\text{-})}(x)} = \frac{\Gamma(nk)}{(\Gamma(k))^n}\frac{\prod_{1\le i<j\le n-1}(y_i-y_j)}{\prod_{1\le i<j\le n}(x_i-x_j)^{2k-1}}\prod_{i=1}^{n-1}\prod_{j=1}^{n}|y_i-x_j|^{k-1}, \tag{4.1}$$

$$\begin{aligned}\frac{J_{\rm o}^{(n,n-1)}(x,y)}{J_{\rm o}^{(\text{-},n-1)}(y)} &= \frac{\Gamma(2+\alpha+\beta+(n-1)k)}{\Gamma(1+\alpha)\Gamma(1+\beta)(\Gamma(k))^{n-1}}\prod_{i=1}^{n}x_i^{\alpha}(1-x_i)^{\beta}\prod_{i=1}^{n-1}y_i^{-(\alpha+k)}(1-y_i)^{-(\beta+k)}\\ &\quad\times\frac{\prod_{1\le i<j\le n}(x_i-x_j)}{\prod_{1\le i<j\le n}(y_i-y_j)^{2k-1}}\prod_{i=1}^{n-1}\prod_{j=1}^{n}|y_i-x_j|^{k-1}\end{aligned} \tag{4.2}$$

where the x's and y's are interlaced according to (3.21). The conditional PDF (4.1) can be recognized as the special case $s_1=s_2=\cdots=s_n=k$ of the conditional density function

$$\frac{\Gamma(s_1+\cdots+s_n)}{\Gamma(s_1)\cdots\Gamma(s_n)}\frac{\prod_{1\le i<j\le n-1}(y_i-y_j)}{\prod_{1\le i<j\le n}(x_i-x_j)^{s_i+s_j-1}}\prod_{i=1}^{n-1}\prod_{j=1}^{n}|y_i-x_j|^{s_j-1} \tag{4.3}$$

appearing in the work of Anderson [2] on the Selberg integral. (Note that the q-generalization of this same density appears in Evans' q-integral (B.5); also we refer to [33] for a further integration method to verify that the normalization is correct.) Here we will show that (4.3), with the s_i non-negative integers or half integers, and thus according to (4.1) the conditional PDF associated to the case $\bar{a}=a_1=0$ of the interpolating ensemble (3.20), can be derived from a random matrix problem.

The random matrix problem relates to the corank 1 random projection of a fixed matrix. Specifically, we seek the eigenvalue PDF of

$$M := \Pi A\Pi, \qquad \Pi := \mathbf{1}-\vec{x}\vec{x}^{\,\dagger} \tag{4.4}$$

where A is a real symmetric, or complex Hermitian, fixed matrix, $\vec{x}$ is a real, or complex, normalized Gaussian column vector of the same number of rows as A and $\mathbf{1}$ denotes the identity matrix. The eigenvalue PDF depends only on the eigenvalues of A, which we take to be $a_1>a_2>\cdots>a_n$ with multiplicities $m_1,m_2,\dots,m_n$.

All but n eigenvalues of (4.4) must coincide with the eigenvalues a_i of A and must occur in M with multiplicity m_i-1. For the latter result we make use of the following formula for the characteristic polynomial of M.

Lemma 1. *We have*

$$\det(M-\lambda\mathbf{1}) = -\lambda\det(A-\lambda\mathbf{1})\mathrm{Tr}\Big((A-\lambda\mathbf{1})^{-1}\vec{x}\vec{x}^{\,\dagger}\Big). \tag{4.5}$$

Proof. Simple manipulation using (4.4) shows

$$\det(M-\lambda\mathbf{1}) = \det(A-\lambda\mathbf{1})\det(\mathbf{1}+(A-\lambda\mathbf{1})^{-1}(-A\vec{x}\vec{x}^{\,\dagger}-\vec{x}\vec{x}^{\,\dagger}A+\vec{x}\vec{x}^{\,\dagger}A\vec{x}\vec{x}^{\,\dagger})).$$

The matrix in the second determinant is of the form $\mathbf{1}+Y$ where Y has rank 1, and in such a circumstance we have $\det(\mathbf{1}+Y)=1+\mathrm{Tr}\,Y$. Using this fact, then further manipulation using $\mathrm{Tr}(\vec{x}\vec{x}^{\,\dagger})=1$ (which in turn follows from the assumption that $\vec{x}$ is normalized) gives (4.5). □

Now (4.5) shows there is an eigenvalue $\lambda=0$, and that the remaining eigenvalues satisfy

$$\prod_{l=1}^{n}(a_l-\lambda)^{m_l}\sum_{i=1}^{n}\frac{\sum_{j=1}^{m_i}u_i^{(j)}}{a_i-\lambda}=0, \qquad w_i := \sum_{j=1}^{m_j}u_i^{(j)} \tag{4.6}$$

where the $u_i^{(j)}$ denote the diagonal elements of $\vec{x}\vec{x}^\dagger$. This shows immediately that M has eigenvalues a_i with multiplicities $m_i - 1$, and the remaining $n-1$ eigenvalues given by the zeros of the random rational function

$$R(\lambda) := \sum_{i=1}^{n} \frac{w_i}{a_i - \lambda}. \tag{4.7}$$

The fact that the w_i are positive (being equal to sums of squares) implies that the roots of $R(\lambda)$ are all real (as must be since M is Hermitian) and further have the interlacing property

$$a_1 > \lambda_1 > a_2 > \lambda_2 > \cdots > \lambda_{n-1} > a_n \tag{4.8}$$

(c.f. (3.21)).

We would like to compute the distribution of the roots of λ for given $a_1, \dots, a_n$ and the $\{w_i\}$ random. This depends crucially on the precise distribution of the $\{w_i\}$. Now w_i has the form

$$w_i = X_i/(X_1 + \cdots + X_n) \tag{4.9}$$

where X_i consists of βm_i ($\beta = 1$ for $\vec{x}$ real and $\beta = 2$ for $\vec{x}$ complex) independent real Gaussians with mean zero and standard deviation σ, and thus has the gamma distribution $\Gamma(s_i, 2\sigma)$, $s_i := \beta m_i/2$. It follows that the PDF for $(w_1, w_2, \dots, w_{n-1}; w_n)$ is equal to the Dirichlet distribution

$$\frac{\Gamma(s_1 + \cdots + s_n)}{\Gamma(s_1)\cdots\Gamma(s_n)} \prod_{i=1}^{n} w_i^{s_i - 1}, \qquad w_n := 1 - \sum_{j=1}^{n-1} w_j, \quad w_j > 0. \tag{4.10}$$

The working in Anderson's paper [2] shows us that the distribution of the roots λ_i of $R(\lambda)$ when the $\{w_i\}$ are distributed according to (4.10) is given by (4.3) with $y_i = \lambda_i$, $x_i = a_i$. As a consequence we can specify the sought eigenvalue distribution.

Corollary 1. *The eigenvalues $\lambda_1, \dots, \lambda_{n-1}$ of M in (4.4) differing from the eigenvalues of A and from 0 have the PDF (4.3) with $y_i = \lambda_i$ $(i = 1, \dots, n-1)$, $x_i = a_i$ $(i = 1, \dots, n)$ and $s_i = \beta m_i/2$ ($\beta = 1$ for $\vec{x}$ real and $\beta = 2$ for $\vec{x}$ complex).*

In the case that all eigenvalues of A are distinct (or doubly degenerate in the case of $\vec{x}$ complex), the eigenvalues of M can be interpreted as so called radial Gelfand-Tzetlin coordinates introduced by Guhr and Kohler [22]. Further, as shown in Appendix C, this observation and Corollary 1 can be used to rederive a recursion formula obtained in [22] for certain matrix Bessel functions.

4.1 Construction of interpolating Jacobi ensembles

We can make use of Corollary 1 to determine explicit random matrices with eigenvalue PDFs which realize (1.4) and (1.5). Consider first (1.4). Essential to our construction are random matrices with a doubly degenerate spectrum which have an eigenvalue PDF of the form (1.9). The required random matrices are known from [7] (see also [16]). Thus consider a member S of the circular ensemble $\mathrm{CSE}_{(n^*+n)}$ (for the definition and construction of such matrices — in which each element is itself the 2×2 matrix representation of a real quaternion — see e.g. [16]). Decompose S as

$$S = \begin{bmatrix} r_{n^* \times n^*} & t'_{n^* \times n} \\ t_{n \times n^*} & r'_{n \times n} \end{bmatrix} \tag{4.11}$$

where $n^* \geq n$ and the subscript on the blocks tells us their dimension (with the already mentioned qualification that each element is a 2×2 matrix). Then we have from [7, 16] that the random matrix $tt^\dagger$

has eigenvalue PDF (1.9) with $a = 2(n^* - n)$. Moreover, if we append to t an extra n_0 rows of zeros and denote this $\tilde{t}$ say, then $\tilde{t}\tilde{t}^\dagger$ has a zero eigenvalue of multiplicity n_0, and n eigenvalues with PDF (1.9) of multiplicity 2. Substituting $\tilde{t}\tilde{t}^\dagger$ for A in (4.4) we can make use of Corollary 1 to deduce that the PDF for the non-zero eigenvalues of M realize (1.4).

Theorem 1. *With $\tilde{t}$ specified above and $\vec{x}$ a normalized complex Gaussian column vector of $2n+n_0$ rows, the non-zero eigenvalues of the random matrix*

$$M = \Pi\tilde{t}\tilde{t}^\dagger\Pi, \qquad \Pi = \mathbf{1} - \vec{x}\vec{x}^\dagger$$

have the PDF (1.9) with

$$a = 2(n^* - n), \qquad A = 2(n^* - n) - 2n_0 + 1. \tag{4.12}$$

Proof. Let the non-zero eigenvalues of $\tilde{t}\tilde{t}^\dagger$ be denoted $a_1, a_2, \dots, a_n$. We know they are doubly degenerate and have distribution (1.9) with a therein given by (4.12), and furthermore we know that $\tilde{t}\tilde{t}^\dagger$ has a zero eigenvalue of multiplicity n_0. According to (4.6) and (4.8) the matrix M then has a zero eigenvalue also of multiplicity n_0, eigenvalues $a_1, \dots, a_n$ with multiplicity 1 and eigenvalues $\lambda_1, \dots, \lambda_n$ such that

$$a_1 > \lambda_1 > a_2 > \cdots > \lambda_n > 0. \tag{4.13}$$

From Corollary 1 the conditional PDF of $\lambda_1, \dots, \lambda_n$ given $a_1, \dots, a_n$ is proportional to

$$\prod_{i=1}^n a_i^{-2}\Big(\frac{\lambda_i}{a_i}\Big)^{n_0-1} \prod_{1\le i<j\le n} \frac{(\lambda_i - \lambda_j)}{(a_i - a_j)^3} \prod_{i,j=1}^n |a_i - \lambda_j|. \tag{4.14}$$

But the PDF of $a_1, \dots, a_n$ is given by (1.9) (with the x's replaced by a's and a given by (4.12)), so forming the product with (4.14) shows the eigenvalue PDF of M is proportional to

$$\prod_{i=1}^n a_i^{2(n^*-n)-1}\Big(\frac{\lambda_i}{a_i}\Big)^{n_0-1} \prod_{1\le i<j\le n} (a_i - a_j)(\lambda_i - \lambda_j) \prod_{i,j=1}^n |a_i - \lambda_j|.$$

After relabelling we recognize this as the PDF (1.4). □

A significant feature of Theorem 1 is that by construction

$$\mathrm{odd}(M) = \mathrm{JSE}_n\Big|_{\substack{a\mapsto a+1\\ b=0}}, \tag{4.15}$$

where odd(M) refers to the distribution of the odd labelled eigenvalues of the random matrix M. Choosing any particular value of A allowed by (4.12) and multiplying both sides of (4.15) by $\prod_{l=1}^n x_{2l-1}^{-A}$ we see that (4.15) implies

$$\mathrm{odd}(\mathrm{JOE}_{2n}\big|_{\substack{a=(c-1)/2\\ b=0}}) = \mathrm{JSE}_n\big|_{\substack{a=c+1\\ b=0}} \tag{4.16}$$

where $c = 2n_0 - 1$. Identities of this type were classified in [18] using functional properties of the PDFs. In fact it was found that (4.16) is one of only two identities relating every second eigenvalue in a matrix ensemble with orthogonal symmetry and an even number of eigenvalues, to a matrix ensemble with symplectic symmetry (for the other see (5.9) below). A challenge was issued to provide a matrix derivation of such results; by way of the above construction this challenge has been answered for the particular identity (4.16).

Let us now seek a realization of (1.5) as an eigenvalue PDF. Guided by the above construction of random matrices with eigenvalue PDF (1.4), we first seek random matrices with a doubly degenerate spectrum which have an eigenvalue PDF equal to (1.12) which is the $A \to -\infty$ limit of (1.4). From [7, 16] we know that with S a $(n^*+n) \times (n^*+n)$ random unitary matrix decomposed as in (4.11), the random matrix $tt^\dagger$ has eigenvalue PDF (1.12) with

$$a = n^* - n, \tag{4.17}$$

although the eigenvalues are all distinct. To obtain a doubly degenerate spectrum with the same eigenvalue PDF, we simply replace each complex element $x+iy$ of t by its 2×2 real matrix representation

$$\begin{bmatrix} x & y \\ -y & x \end{bmatrix}. \tag{4.18}$$

To this doubly degenerate spectrum with eigenvalue PDF (1.12) we want to add a zero eigenvalue of degeneracy n_0. As noted below (4.11), this is achieved by simply appending n_0 rows of zeros; let us denote the real representation of t so modified by $\hat{t}$. The real symmetric matrix $\hat{t}\hat{t}^T$ then has a zero eigenvalue of multiplicity n_0 and n eigenvalues with PDF (1.12) of multiplicity 2. We can now use Corollary 1 to obtain the sought realization of (1.5).

Theorem 2. *With $\hat{t}$ specified above and $\vec{x}$ a normalized real Gaussian vector of $2n+n_0$ rows, the non-zero eigenvalues of the random matrix*

$$M = \Pi \hat{t}\,\hat{t}^T \Pi, \qquad \Pi = \mathbf{1} - \vec{x}\vec{x}^T$$

have PDF (1.4) with

$$a = n^* - n, \qquad A = n^* - n - n_0 + 1.$$

Proof. Following the reasoning of the proof of Theorem 1, the matrix M has a zero eigenvalue of multiplicity n_0, a distinct copy of the non-zero eigenvalues $a_1, \dots, a_n$ say of $\hat{t}\hat{t}^T$, and eigenvalues $\lambda_1, \dots, \lambda_n$ satisfying the interlacing condition (4.13). Corollary 1 with $n \mapsto n+1$, $a_{n+1} = 0$, $s_{n+1} = n_0/2$, $s_i = 1$ $(i = 1, \dots, n)$ gives that the conditional PDF of $\lambda_1, \dots \lambda_n$ given $a_1, \dots, a_n$ is proportional to

$$\prod_{i=1}^n a_i^{-1} \Big(\frac{\lambda_i}{a_i} \Big)^{n_0/2-1} \prod_{1 \le i < j \le n} \frac{(\lambda_i - \lambda_j)}{(a_i - a_j)} \prod_{i,j=1}^n |a_i - \lambda_j|.$$

The eigenvalue PDF of M now follows by multiplying this by the PDF of $a_1, \dots, a_n$ as given by (1.12) (with the x's replaced by a's). After relabelling the coordinates the PDF (1.5) results with the parameters as stated. □

Analogous to (4.15), by construction

$$\mathrm{odd}(M) = \mathrm{JUE}_n \Big|_{b=0}. \tag{4.19}$$

Since with $A = 0$ the eigenvalue PDF of M coincides with that of the matrix ensemble $\mathrm{JOE}_n|_{\substack{a \mapsto (a-1)/2 \\ b=0}} \cup \mathrm{JOE}_n|_{\substack{a \mapsto (a-1)/2 \\ b=0}}$ (recall sentence below (1.10)) we have a matrix theoretic understanding of the relation [18]

$$\mathrm{odd}\Big(\mathrm{JOE}_n|_{\substack{a \mapsto (a-1)/2 \\ b=0}} \cup \mathrm{JOE}_n|_{\substack{a \mapsto (a-1)/2 \\ b=0}} \Big) = \mathrm{JUE}_n \Big|_{b=0}. \tag{4.20}$$

4.2 A random three term recurrence for interpolating Jacobi ensembles

In this section, inspired by the recent work [12], it will be shown that $J_{\rm o}^{(n,n-1)}(x,y)$, specified by (3.24), and $J_{\rm e}^{(n,n)}(x,y)$, specified by (3.29), can be sampled from the zeros of a polynomial which in turn is specified using a random three term recurrence. Consider first (3.20). We begin by noting that (4.2), like (4.1) is intimately related to the Anderson density (4.3). Thus in the latter put $n \mapsto n+1$, relabel the x's by y's and the y's by x's, then set $y_1 = 1$, $y_{n+1} = 0$, relabel y_{i+1} by y_i $(i = 1, \dots, n-1)$ and put $s_1 = \beta + 1$, $s_{n+1} = \alpha + 1$ and $s_i = k$ $(i = 2, \dots, n)$ to obtain (4.2). As a consequence, Anderson's result relating the random rational function (4.7), with coefficients distributed according to the Dirichlet distribution (4.10), to (4.3) tells us we can similarly specify a random rational function related to (4.2).

Corollary 2. *Denote the Dirichlet distribution (4.10) by $D_n[s_1, \dots, s_{n-1}; s_n]$. Let $(w_0, \dots, w_{n-1}; w_n)$ be distributed according to $D_{n+1}[\beta + 1, (k)^{n-1}; \alpha + 1]$, where the notation $(k)^{n-1}$ denotes k repeated $n-1$ times. We have that the roots of the random rational function*

$$\tilde{R}_{n+1}(x) := \frac{w_0}{x-1} + \frac{w_n}{x} + \sum_{i=1}^{n-1} \frac{w_i}{x - y_i} \tag{4.21}$$

are distributed according to the PDF (4.2).

Anderson's result stated below (4.10) and Corollary 2 can be used to derive a random three term recurrence which specifies a polynomial, the zeros of which sample from the joint PDF (3.24), and also sample from the marginal PDF (3.26). We will specify the recurrence by first detailing how it leads to a polynomial with the sought properties in the low degree cases, before stating its general form.

`Step 1` Consider (4.21) in the case $n = 1$ and write $w_0 \mapsto w_0^{(1)}$, $w_1 \mapsto w_1^{(1)}$. Let $(w_0^{(1)}; w_1^{(1)})$ be distributed according to $D_2[\beta^{(1)} + 1; \alpha^{(1)} + 1]$. Let $\lambda_1^{(1)}$ denote the zero of (4.21) in this case and form the polynomial

$$A_1(x) := x - \lambda_1^{(1)}. \tag{4.22}$$

It follows from Corollary 2 that $\lambda_1^{(1)}$ is distributed according to

$$\frac{J_{\rm o}^{(1,0)}(x,y)}{J_{\rm o}^{(\cdot,0)}(x)}\Big|_{\alpha=\alpha^{(1)},\,\beta=\beta^{(1)}} =: P(\lambda_1^{(1)}) \tag{4.23}$$

which is itself the Dirichlet distribution $D_2[\alpha^{(1)} + 1; \beta^{(1)} + 1]$.

`Step 2` Define $A_1(x)$ by (4.22) and also define

$$A_0(x) := 1. \tag{4.24}$$

Let $(w_0^{(2)}, w_1^{(2)}; w_2^{(2)})$ be distributed according to $D_3[\beta^{(2)} + 1, k; \alpha^{(2)} + 1]$ and construct the random quadratic polynomial

$$A_2(x) := w_2^{(2)}(x-1)A_1(x) + w_0^{(2)} x A_1(x) + w_1^{(2)} x(x-1)A_0(x). \tag{4.25}$$

Dividing both sides by $x(x-1)A_1(x)$, this reads

$$\frac{A_2(x)}{x(x-1)A_1(x)} = \frac{w_2^{(2)}}{x} + \frac{w_0^{(2)}}{x-1} + \frac{w_1^{(2)}}{x - \lambda_1^{(1)}}.$$

Because $\sum_{\mu=0}^{2} w_\mu^{(2)} = 1$ and the $w_\mu^{(2)}$ are positive, $A_2(x)$ must be monic with real roots and we write

$$A_2(x) = (x - \lambda_1^{(2)})(x - \lambda_2^{(2)}).$$

It follows from Corollary 2 that the conditional distribution of $\{\lambda_1^{(2)}, \lambda_2^{(2)}\}$ given $\lambda_1^{(1)}$ has the form

$$\frac{J_{\rm o}^{(2,1)}(x,y)}{J_{\rm o}^{(\cdot,1)}(y)}\Big|_{\alpha=\alpha^{(2)},\,\beta=\beta^{(2)}} =: P(\lambda_1^{(2)}, \lambda_2^{(2)} | \lambda_1^{(1)}) \tag{4.26}$$

and so the joint density of $\lambda_1^{(1)}, \lambda_1^{(2)}, \lambda_2^{(2)}$ is

$$P(\lambda_1^{(1)}) P(\lambda_1^{(2)}, \lambda_2^{(2)} | \lambda_1^{(1)}). \tag{4.27}$$

If we set

$$\alpha^{(1)} = \alpha^{(2)} + \alpha_1 + k, \quad \beta^{(1)} = \beta^{(2)} + \beta_1 + k \tag{4.28}$$

we recognize this as the joint distribution function $J_{\rm o}^{(2,1)}(x,y)$ with $\alpha = \alpha^{(2)}, \beta = \beta^{(2)}$. According to (3.26), if we now set $\alpha_1 = \beta_1 = 0$ we can compute the marginal distribution of $\lambda_1^{(2)}, \lambda_2^{(2)}$,

$$\int_{\lambda_1^{(2)} > \lambda_1^{(1)} > \lambda_2^{(2)}} d\lambda_1^{(1)} \, P(\lambda_1^{(1)}) P(\lambda_1^{(2)}, \lambda_2^{(2)} | \lambda_1^{(1)}) \Big|_{\alpha_1 = \beta_1 = 0} = J_{\rm o}^{(2,\text{-})}(\lambda_1^{(2)*}, \lambda_2^{(2)*}) \Big|_{\substack{\alpha = \alpha^{(2)} \\ \beta = \beta^{(2)}}}, \tag{4.29}$$

where the use of the $*$ on the right hand side indicates the parameters have been chosen so that (4.28) holds with $\alpha_1 = \beta_1 = 0$. This distribution is realized by the roots of $A_2^*(x)$ when $\lambda_1^{(1)}$ is not observed.

To proceed further requires an extension of the above arguments. Let us consider

$$\frac{A_1(x)}{A_2^*(x)} = \sum_{l=1}^{2} \frac{u_l}{x - \lambda_2^{(l)*}}, \qquad u_1 + u_2 = 1 \tag{4.30}$$

and pose the question as to what distribution of $(u_1; u_2)$ is required so that the distribution of the zero on the right hand side has the same distribution as $\lambda_1^{(1)}$, with $\lambda_1^{(2)*}, \lambda_2^{(2)*}$ distributed by the right hand side of (4.29), and is thus specified by (4.23)?

According to Anderson's result stated below (4.10) we have that with $(u_1; u_2)$ distributed according to $D_2(k; k)$, the root of (4.30) has conditional distribution

$$\frac{J_{\rm o}^{(2,1)}(\lambda_1^{(2)*}, \lambda_2^{(2)*}; \lambda_1^{(1)}) \Big|_{\alpha_1 = \beta_1 = 0}}{J_{\rm o}^{(2,\text{-})}(\lambda_1^{(2)*}, \lambda_2^{(2)*})}.$$

Thus the corresponding marginal distribution of $\lambda_1^{(1)}$ is

$$\begin{aligned} &\int_{1 > \lambda_1^{(2)} > \lambda_1^{(1)} > \lambda_2^{(2)} > 0} d\lambda_1^{(2)} d\lambda_2^{(2)} \, J_{\rm o}^{(2,\text{-})}(\lambda_1^{(2)*}, \lambda_2^{(2)*}) \Big|_{\substack{\alpha = \alpha^{(2)} \\ \beta = \beta^{(2)}}} \frac{J_{\rm o}^{(2,1)}(\lambda_1^{(2)}, \lambda_2^{(2)}; \lambda_1^{(1)}) \Big|_{\alpha_1 = \beta_1 = 0}}{J_{\rm o}^{(2,\text{-})}(\lambda_1^{(2)*}, \lambda_2^{(2)*})} \\ &\quad = J_{\rm o}^{(\text{-},1)}(\lambda_1^{(1)}) \Big|_{\substack{\alpha = \alpha^{(2)} + k \\ \beta = \beta^{(2)} + k}} \end{aligned}$$

where use has been made of (3.25) and (4.28). This is indeed the same distribution as (4.23), provided we set $\alpha^{(1)} = \alpha^{(2)} + k$, $\beta^{(1)} = \beta^{(2)} + k$ therein. Let us denote $A_1(x)$ with the parameters so specialized by $A_1^{\#}(x)$.

As well as making use of (4.30) with (u_1, u_2) distributed according to $D_2[k; k]$, we require some special properties of the Dirichlet and beta distributions. First we recall that the Dirichlet distribution $D_2[\alpha; \beta]$ and the beta distribution $B[\alpha, \beta]$ are the same thing. We require the fact that if $(w_0, \dots, w_{n-1}; w_n)$ is distributed according to $D_{n+1}[\alpha_0, \dots, \alpha_{n-1}; \alpha_n]$, then the marginal distribution of w_j $(j = 0, \dots, n-1)$ is given by $B[\alpha_j, \sum_{i=0, i \neq j}^n \alpha_i]$ and the marginal distribution of $w_j + w_k$, $(j \neq k, \, j, k \leq n)$ is $B[\alpha_j + \alpha_k, \sum_{i=0, i \neq j,k}^n \alpha_i]$. We also require the property of the beta distribution (see e.g. [32, pg. 42])

$$B[a + b, c] B[a, b] = B[a, b + c] \tag{4.31}$$

where here — in an abuse of notation — the left hand side means the product of random variables from the respective distributions, and the right hand side tells us the distribution of the product.

`Step 3` Analogous to the construction of $A_2(x)$, we construct $A_3(x)$ by the random three term recurrence

$$A_3(x) := w_2^{(3)}(x-1)A_2^*(x) + w_0^{(3)}xA_2^*(x) + w_1^{(3)}x(x-1)A_1^{\#}(x)$$

where $(w_0^{(3)}, w_1^{(3)}; w_2^{(3)})$ is distributed according to $D_3[\beta^{(3)}+1, 2k; \alpha^{(3)}+1]$, or equivalently

$$\frac{A_3(x)}{x(x-1)A_2^*(x)} = \frac{w_2^{(3)}}{x} + \frac{w_0^{(3)}}{x-1} + w_1^{(3)}\frac{A_1^{\#}(x)}{A_2^*(x)}.$$

For $A_1^{\#}(x)/A_2^*(x)$ we substitute (4.30). Now the theory above (4.31) tells us that the marginal distribution of $w_1^{(3)}$ is $B[2k, \alpha^{(3)}+\beta^{(3)}+2]$, while the distribution of u_1 in (4.30) is $B[k,k]$. Applying (4.31) it follows that we can write

$$w_1^{(3)}\frac{A_1^{\#}(x)}{A_2^*(x)} = \sum_{l=1}^{2}\frac{\tilde{u}_l}{x-\lambda_2^{(l)*}}$$

where $\tilde{u}_1$ has distribution $B[k, \alpha^{(3)}+\beta^{(3)}+k+2]$ and $\tilde{u}_1+\tilde{u}_2$ has distribution $B[2k; \alpha^{(3)}+\beta^{(3)}+2]$. Consequently we have

$$\frac{A_3(x)}{x(x-1)A_2^*(x)} = \frac{\tilde{w}_3^{(3)}}{x} + \frac{\tilde{w}_0^{(3)}}{x-1} + \sum_{l=1}^{2}\frac{\tilde{w}_l^{(3)}}{x-\lambda_2^{(l)*}} \tag{4.32}$$

where $(\tilde{w}_0^{(3)}, \tilde{w}_1^{(3)}, \tilde{w}_2^{(3)}; \tilde{w}_3^{(3)})$ has distribution $D_4[\beta^{(3)}+1, (k)^2; \alpha^{(3)}+1]$. Arguing now as in the derivation of (4.26) that

$$A_3(x) = (x-\lambda_1^{(3)})(x-\lambda_2^{(3)})(x-\lambda_3^{(3)})$$

where the conditional distribution of $\{\lambda_1^{(3)}, \lambda_2^{(3)}, \lambda_2^{(3)}\}$ given $\{\lambda_1^{(2)*}, \lambda_2^{(2)*}\}$ has the form

$$\frac{J_{\rm o}^{(3,2)}(x,y)}{J_{\rm o}^{(\text{-},2)}(y)}\Big|_{\alpha=\alpha^{(3)},\beta=\beta^{(3)}} =: P(\lambda_1^{(3)}, \lambda_2^{(3)}, \lambda_2^{(3)}|\lambda_1^{(2)*}, \lambda_2^{(2)*}).$$

The joint density of $\{\lambda_1^{(2)*}, \lambda_2^{(2)*}, \lambda_1^{(3)}, \lambda_2^{(3)}, \lambda_2^{(3)}\}$ is therefore

$$J_{\rm o}^{(2,\text{-})}(\lambda_1^{(2)*}, \lambda_2^{(2)*})P(\lambda_1^{(3)}, \lambda_2^{(3)}, \lambda_2^{(3)}|\lambda_1^{(2)*}, \lambda_2^{(2)*}) \tag{4.33}$$

which with

$$\alpha^{(2)} = \alpha^{(3)} + \alpha_1 + k, \quad \beta^{(2)} = \beta^{(3)} + \beta_1 + k$$

we recognize as the joint distribution function $J_{\rm o}^{(3,2)}(x,y)$ with $\alpha = \alpha^{(3)}$, $\beta = \beta^{(3)}$. As with (4.29), the marginal distribution of $\lambda_1^{(3)}, \lambda_2^{(3)}, \lambda_3^{(3)}$ can be computed in the case $\alpha_1 = \beta_1 = 0$. Thus it follows from (3.26) that

$$\begin{aligned}\int_{\lambda_1^{(3)}>\lambda_1^{(2)}>\lambda_2^{(3)}>\lambda_2^{(2)}>\lambda_3^{(3)}} & d\lambda_1^{(2)}d\lambda_2^{(2)}\, J_{\rm o}^{(2,\text{-})}(\lambda_1^{(2)*}, \lambda_2^{(2)*})P(\lambda_1^{(3)}, \lambda_2^{(3)}, \lambda_3^{(3)}|\lambda_1^{(2)*}, \lambda_2^{(2)*})\Big|_{\alpha_1=\beta_1=0} \\ &= J_{\rm o}^{(3,\text{-})}(\lambda_1^{(3)*}, \lambda_2^{(3)*}, \lambda_3^{(3)*}),\end{aligned} \tag{4.34}$$

giving the distribution of the zeros of $A_3^*(x)$ when $\lambda_1^{(2)*}, \lambda_2^{(2)*}$ are not observed.

Step 3 is representative of the general step n in generating the recurrence. Of course we now need inductive hypotheses relating to the roots of polynomials generated in earlier steps. In particular, we suppose that in step $n-2$ a polynomial $A_{n-2}(x)$ has been generated and the density of its roots is given by $J^{(\text{-},n-2)}(y)|_{\alpha_1=\beta_1=0}$ as specified by (3.25) with $\alpha = \alpha^{(n-2)}$, $\beta = \beta^{(n-2)}$. With the special choice of parameters $\alpha^{(n-2)} = \alpha^{(n-1)} + k, \beta^{(n-2)} = \beta^{(n-1)} + k$ we denote $A_{n-2}(x)$ by $A_{n-2}^{\#}(x)$. At step $n-1$

we require that a polynomial $A^*_{n-1}(x)$ has been generated which has the density of its roots given by $J_{\rm o}^{(n-1,-)}(x)$ as specified by (3.26) with $\alpha=\alpha^{(n-1)}$, $\beta=\beta^{(n-1)}$.

`Step n` We construct $A_n(x)$ by the random three term recurrence

$$A_n(x)=w_2^{(n)}(x-1)A^*_{n-1}(x)+w_0^{(n)}xA^*_{n-1}(x)+w_1^{(n)}x(x-1)A^{\#}_{n-2}(x) \tag{4.35}$$

where $(w_0^{(n)},w_1^{(n)},w_2^{(n)})$ is distributed according to $D_3[\beta^{(n)}+1,(n-1)k;\alpha^{(n)}+1]$. Arguing as in the derivation of (4.33) we see that with

$$\alpha^{(n-1)}=\alpha^{(n)}+\alpha_1+k, \qquad \beta^{(n-1)}=\beta^{(n)}+\beta_1+k \tag{4.36}$$

the joint distribution of the roots of $A_n(x)$ and $A^*_{n-1}(y)$ is given by $J_{\rm o}^{(n,n-1)}(x,y)$ with $\alpha=\alpha^{(n)}$, $\beta=\beta^{(n)}$, and that with $\alpha_1=\beta_1=0$ the marginal distribution of the roots of $A_n(x)$ is given by $J_{\rm o}^{(n,-)}(x)$ with $\alpha=\alpha^{(n)}$, $\beta=\beta^{(n)}$.

From a practical point of view, our objective is to sample from $J_{\rm o}^{(n,n-1)}(x,y)$ and $J_{\rm o}^{(n,-)}(x)$ for a fixed value of n and fixed parameters. To sample from $J_{\rm o}^{(n,-)}(x)$ with $\alpha=\alpha_0$, $\beta=\beta_0$ we implement the above steps with

$$\alpha^{(j)}=(n-j)k+\alpha_0, \qquad \beta^{(j)}=(n-j)k+\beta_0. \tag{4.37}$$

We see that in this situation $A^{\#}_j(x)=A^*_j(x)$ and so $\{A^{\#}_j(x)\}_{j=2,\dots,n}$ is determined by the random recurrence

$$A^{\#}_j(x)=w_2^{(j)}(x-1)A^{\#}_{j-1}(x)+w_0^{(j)}xA^{\#}_{j-1}(x)+w_1^{(j)}x(x-1)A^{\#}_{j-2}(x) \tag{4.38}$$

where $(w_0^{(j)},w_1^{(j)},w_2^{(j)})$ is distributed according to $D_3[(n-j)k+\beta_0+1,(j-1)k;(n-j)k+\alpha_0+1]$. The initial conditions for the recurrence are $A^{\#}_{-1}(x)=0$ and $A^{\#}_0(x)=1$. The zeros of $A^{\#}_n(x)$ then are distributed according to $J_{\rm o}^{(n,-)}(x)|_{\substack{\alpha=\alpha_0\\ \beta=\beta_0}}$.

If our objective is to sample from $J_{\rm o}^{(n,n-1)}(x,y)$ with $\alpha=\alpha_0$, $\beta=\beta_0$, we again compute $\{A^{\#}_j(x)\}_{j=0,\dots,n-1}$ this time replacing α_0, β_0 by $\alpha_0+\alpha_1$, $\beta_0+\beta_1$ throughout. Let us write $A^{\#}_j(x)$ with these parameters as $\tilde{A}^{\#}_j(x)$. Because we now have $\alpha^{(n-1)}=k+\alpha_0+\alpha_1$, $\beta^{(n-1)}=k+\beta_0+\beta_1$ we see that $A^*_{n-1}(x)=\tilde{A}^{\#}_{n-1}(x)$, so according to (4.35) the final step is to compute

$$A_n(x)=w_2^{(n)}(x-1)\tilde{A}^{\#}_{n-1}(x)+w_0^{(n)}x\tilde{A}^{\#}_{n-1}(x)+w_1^{(n)}x(x-1)\tilde{A}^{\#}_{n-2}(x) \tag{4.39}$$

where $(w_0^{(n)},w_1^{(n)},w_2^{(n)})$ is distributed according to $D_3[\beta_0+1,(n-1)k;\alpha_0+1]$. We then have that the zeros of $(A_n(x),\tilde{A}^{\#}_{n-1}(y))$ have the joint distribution $J_{\rm o}^{(n,n-1)}(x,y)$.

Let us now turn our attention to sampling from $J_{\rm e}^{(n,n)}(x,y)$ as specified by (3.29). First we note from Anderson's result stated below (4.10) that the random rational function

$$\hat{R}_{n+1}(x):=\frac{w_{n+1}}{x}+\sum_{i=1}^n\frac{w_i}{x-y_i}, \tag{4.40}$$

where $(w_1,\dots,w_n;w_{n+1})$ is distributed according to $D_{n+1}[(k)^n;\alpha+1]$, has the PDF for its zeros given by $J_{\rm e}^{(n,n)}(x,y)/J_{\rm e}^{(-,n)}(y)$. Let us define $\{A^{\#{\rm e}}_j(x)\}_{j=0,\dots,n}$ as specified by the recurrence (4.38) but with $\alpha_0\mapsto\alpha_0+\alpha_1+k$, $\beta_0\mapsto\beta_1$ throughout. Furthermore, with (w_1,w_2) distributed according to $B_n[nk,\alpha_0+1]$ define

$$V_n(x)=w_2A^{\#{\rm e}}_n(x)+w_1xA^{\#{\rm e}}_{n-1}(x). \tag{4.41}$$

The significance of $V_n(x)$ is seen by noting from the argument below (4.30) that with $y_1,\dots,y_n$ denoting the zeros of $A^{\#{\rm e}}_j(x)$ we have

$$\frac{A^{\#{\rm e}}_{n-1}(x)}{A^{\#{\rm e}}_n(x)}=\sum_{l=1}^n\frac{u_l}{x-y_l}$$

where $(u_1,\dots,u_n)$ is distributed according to $D_n[(k)^{n-1};k]$, and then proceeding as in the derivation of (4.31) to deduce from this the expansion

$$\frac{V_n(x)}{xA_n^{\#\mathrm{e}}(x)} = \frac{\tilde{w}_{n+1}}{x} + \sum_{l=1}^{n} \frac{\tilde{w}_l}{x-y_l^{(j)}} \tag{4.42}$$

where $(\tilde{w}_1,\dots,\tilde{w}_n;\tilde{w}_{n+1})$ is distributed according to $D_{n+1}[(k)^n;\alpha_0+1]$. The right hand side of (4.42) is just the rational function (4.40), and so the PDF for its zeros, given $\{y_1,\dots,y_n\}$, is

$$\frac{J_{\mathrm{e}}^{(n,n)}(x,y)}{J_{\mathrm{e}}^{(\text{-},n)}(y)}\Big|_{\alpha=\alpha_0}. \tag{4.43}$$

But the marginal distribution of $\{y_1,\dots,y_n\}$ is $J_{\mathrm{o}}^{(n,\text{-})}(y)$ with $\alpha=\alpha_0+\alpha_1+k$, $\beta=\beta_1$. Multiplying this by (4.43) shows that the joint distribution of the zeros of $\{V_n(x),A_n^{\#\mathrm{e}}(y)\}$ is given by $J_{\mathrm{e}}^{(n,n)}(x,y)|_{\alpha=\alpha_0}$.

We note from (3.31) that if we set $\alpha_1=0$, $\beta_1=\beta_0-k$ in the construction of $\{A_j^{\#\mathrm{e}}(x)\}_{j=0,\dots,n}$ and then compute $V_n(x)$ according to (4.41), the marginal distribution of the zeros of $V_n(x)$ are given by $J_{\mathrm{e}}^{(n,\text{-})}(x)|_{\substack{\alpha=\alpha_0\\ \beta_1=\beta_0-k}}$. But according to (3.31) and (3.26) the latter is identical to $J_{\mathrm{o}}^{(n,\text{-})}(x)|_{\substack{\alpha=\alpha_0\\ \beta=\beta_0}}$ and so the marginal distribution of the zeros of $V_n(x)$ in this case is the same as that for the zeros of $A_n^{\#}(x)$. Thus with (w_1,w_2) as in (4.41) we have

$$A_n^{\#}(x) = \Big(w_2A_n^{\#}(x) + w_1xA_{n-1}^{\#}(x)\Big)\Big|_{\substack{\alpha_0\mapsto\alpha_0+k\\ \beta_0\mapsto\beta_0-k}} \tag{4.44}$$

The random recurrences (4.38) and (4.44) assume definite forms if we write $\alpha_0=ak$, $\beta_0=bk$ and take the limit $k\to\infty$. Thus the random variables $(w_0^{(j)},w_1^{(j)},w_2^{(j)})$ in (4.38) crystallize to the definite value $((n-j+b)/d,(j-1)/d,(n-j+a)/d)$ where $d=2n-j-1+a+b$ and so (4.38) reads

$$(2n-j-1+a+b)A_j^{\#}(x) = (n-j+a)(x-1)A_{j-1}^{\#}(x)+(n-j+b)xA_{j-1}^{\#}(x)+(j-1)x(x-1)A_{j-2}^{\#}(x) \tag{4.45}$$

with initial conditions $A_{-1}^{\#}(x)=0$, $A_0^{\#}(x)=1$. Similarly (4.44) reads

$$(a+n)A_n^{\#}(x) = \Big(aA_n^{\#}(x)+nxA_{n-1}^{\#}(x)\Big)\Big|_{\substack{a\mapsto a+1\\ \beta_0\mapsto b-1}}. \tag{4.46}$$

Using standard Jacobi polynomial recurrences we can show that the solution of (4.45) is given by

$$A_j^{\#}(x) = \tilde{P}_j^{(n+a-1-j,n+b-1-j)}(x), \tag{4.47}$$

where the use of ˜ indicates the monic version of the corresponding polynomial. Furthermore, (4.47) satisfies (4.46). The fact from (4.47) that $A_n^{\#}(x)=\tilde{P}_n^{(a-1,b-1)}(x)$ can be anticipated. Thus in general the PDF for the marginal distribution of the zeros of $A_n^{\#}(x)$ is given by $J_{\mathrm{o}}^{(n,\text{-})}(x)$, and if we put $\alpha_0=ak$, $\beta_0=bk$ and take the limit $k\to\infty$ it is a known result [35] that this PDF crystallizes at the zeros of the Jacobi polynomial $P_n^{(a-1,b-1)}(x)$.

In Appendix D we make use of our ability to sample from $J_0^{(n,\text{-})}$ to give a Monte Carlo evaluation of a multidimensional integral formula [17] for the bulk two-point correlation function of matrix ensembles with symplectic symmetry.

5 Random matrix realizations of the Laguerre interpolating ensembles

Consider the Dirichlet distribution (4.10). Let $s_n=L$ and scale $w_1,w_2,\dots,w_{n-1}$ so that $w_i\mapsto w_i/L$, $(i=1,\dots,n-1)$. Then in the limit $L\to\infty$ (4.10) reduces to the product of independent gamma distributions

$$\frac{\sigma^{-(n-1)}}{\Gamma(s_1)\cdots\Gamma(s_{n-1})}\prod_{i=1}^{n-1}(w_i/\sigma)^{s_i-1}e^{-w_i/\sigma}, \quad w_i>0 \tag{5.1}$$

with $\sigma=1$. If we also scale $a_i\mapsto a_i/L$ $(i=1,\dots,n-1)$, $\lambda\mapsto\lambda/L$ and set $a_n=1$ then the limiting form of the random rational function (4.7) reads

$$R^{\rm L}(\lambda):=1+\sum_{i=1}^{n-1}\frac{w_i}{a_i-\lambda} \tag{5.2}$$

where the w_i are distributed according to (5.1) (the superscipt "L" denotes Laguerre). The distribution of the roots of (5.2) is given by the appropriate limiting form of the Anderson density (4.3), in accordance with Anderson's result stated below (4.10). Let us take note of the explicit form.

Corollary 3. *Consider the random rational function (5.2) with the coefficients $w_1,\dots,w_{n-1}$ distributed according to (5.1). This has exactly $n-1$ roots, which since the w_i are positive, are real. For given $\{a_i\}$ these roots have the PDF*

$$\frac{1}{\Gamma(s_1)\cdots\Gamma(s_{n-1})}e^{-\sum_{j=1}^{n-1}(\lambda_j-a_j)}\prod_{1\le i<j\le n-1}\frac{(\lambda_i-\lambda_j)}{(a_i-a_j)^{s_i+s_j-1}}\prod_{\substack{i,j=1\\ i\ne j}}^{n}|\lambda_i-a_j|^{s_j-1} \tag{5.3}$$

where

$$\lambda_1>a_1>\lambda_2>a_2>\cdots>\lambda_n>a_n. \tag{5.4}$$

We remark that this PDF is implicit in the work of Evans [14], who was studying Laguerre analogues of the Selberg integral using the method of Anderson. Also, special cases of Corollary 3 are known from [1].

A matrix structure for which the eigenvalue condition reduces to the calculation of the roots of (5.2) (with n replaced by $n+1$ for convenience) is easy to specify. One approach would be to consider the appropriate limiting form of (4.4). Alternatively we can write down the required matrix and check directly that it has the sought property. Thus let A be a real symmetric (complex Hermitian) matrix with eigenvalues $a_1>a_2>\cdots>a_n$ of multiplicities $m_1,\dots,m_n$ respectively. Let X be a vector of independent real standard Gaussians (complex Gaussians) having the same number of rows as A, and consider the matrix

$$M=A+b\vec{x}\vec{x}^\dagger, \qquad b>0. \tag{5.5}$$

A simple calculation along the lines of Lemma 1 and the derivation of (4.4) shows that M has eigenvalues a_i with multiplicities m_i-1, and the remaining n eigenvalues are given by the zeros of the rational function (5.2) (with $n\mapsto n+1$). The w_i in the latter are distributed according to (5.1), with $\sigma=2b/\beta$ and and $s_i=\beta m_i/2$ where $\beta=1$ (real case), $\beta=2$ (complex case). Thus (5.3) with $a_i\mapsto\beta a_i/2b$, $\lambda_i\mapsto\beta\lambda_i/2b$, gives the eigenvalue PDF of M.

The eigenvalue PDF of (5.5) in the complex case with all eigenvalues a_i distinct can be derived in a different way which has the advantage of applying to a more general matrix structure. This is given in Appendix E.

5.1 Construction of interpolating Laguerre matrix ensembles

We can use knowledge of the eigenvalue PDF of (5.5) to construct matrix ensembles which have as their eigenvalue PDF (1.1) and (1.2). For the parameter dependent PDF (1.1) we begin by recalling (see e.g. [16]) that (1.8) is realized as the eigenvalue distribution of the matrix ensemble of $4n\times 4n$

antisymmetric Gaussian random matrices, in which the elements are pure imaginary numbers with each 2×2 block having a real quaternion structure, and one changes variables $\lambda_i^2 \mapsto \lambda_i$. Such matrices are equivalent to block matrices of the form

$$\begin{bmatrix} 0_{2n\times 2n} & X_{2n\times 2n} \\ X^\dagger_{2n\times 2n} & 0_{2n\times 2n} \end{bmatrix} \tag{5.6}$$

where X is an antisymmetric Gaussian complex matrix, real and imaginary parts having variance 1/2, so $X^\dagger X$ has eigenvalue PDF (1.8) with each eigenvalue doubly degenerate.

Theorem 3. *Let X be a $2n \times 2n$ antisymmetric complex Gaussian matrix, and let $\vec{x}$ be a $2n \times 1$ complex Gaussian vector, where the real and imaginary parts of the complex Gaussians have variance 1/2. The random matrix*

$$M = X^\dagger X + b\vec{x}\vec{x}^\dagger \tag{5.7}$$

has eigenvalue PDF (1.1) with $A = 1 - 2/b$.

Proof. The eigenvalues of $X^\dagger X$ have multiplicity 2, and thus according to the result noted below (5.5) the eigenvalues of M consist of the eigenvalues of $X^\dagger X$ with multiplicity 1 ($y_1, \dots, y_n$ say), as well as n additional eigenvalues $x_1, \dots, x_n$ which must satisfy the interlacing condition

$$x_1 > y_1 > x_2 > y_2 > \cdots > x_n > y_n. \tag{5.8}$$

Setting $m_i = 2, \beta = 2$ in the result noted below (5.5), it follows from (5.3) that the conditional distribution of the x's given the y's is proportional to

$$e^{-\sum_{i=1}^n (x_i - y_i)/b} \prod_{1 \le i < j \le n} \frac{(x_i - x_j)}{(y_i - y_j)^3} \prod_{i,j=1}^n |x_i - y_j|.$$

Multiplying this by (1.8) (with the x's relabelled y's) gives that the eigenvalue PDF of M is proportional to

$$e^{-\sum_{i=1}^n (y_i + (x_i - y_i)/b)} \prod_{1 \le i < j \le n} (x_i - x_j)(y_i - y_j) \prod_{i,j=1}^n |x_i - y_j|,$$

and relabelling the eigenvalues gives the desired result. □

By construction

$$\mathrm{even}(M) = \mathrm{LSE}_n|_{a=0}$$

where even(M) refers to the distribution of the even labelled eigenvalues of the random matrix M. The result holds independent of the parameter b in (5.7). Because Theorem 3 also tells us that when $b = 2$ the random matrix M has the same PDF as $\mathrm{LOE}_{2n}|_{a=0}$ matrices, we thus have a matrix theoretic understanding of the relation

$$\mathrm{even}(\mathrm{LOE}_{2n}|_{a=0}) = \mathrm{LSE}_n|_{a=0}. \tag{5.9}$$

We remark that with $\tilde{X}$ denoting the $(2n+1) \times 2n$ random matrix which is obtained from X in (5.7) by adjoining an extra row $\sqrt{b}\vec{x}$, we have

$$\tilde{X}^\dagger \tilde{X} = X^\dagger X + b\vec{x}\vec{x}^\dagger. \tag{5.10}$$

Thus we can interpret Theorem 3 as applying to the square of the singular values of $\tilde{X}$. This latter viewpoint indicates a special property of the case $b = 1$. Then $\tilde{X}$ is identical to the $(2n+1) \times (2n+1)$ version of X with the last column removed. But it is a standard result that the singular values of a matrix interleave with those of a matrix obtained by removing a single row or column (starting with the largest singular value of the larger matrix). Now we know that a $(2n+1) \times (2n+1)$ antisymmetric complex Gaussian matrix is such that $\tilde{X}^\dagger \tilde{X}$ has one zero eigenvalue, and n doubly degenerate eigenvalues, the latter having PDF $\mathrm{LSE}_n|_{a=2}$. The interlacing property now implies

$$\mathrm{odd}(M|_{b=1}) = \mathrm{LSE}_n|_{a=2}. \tag{5.11}$$

As done in [19], this can be checked directly from the PDF (1.1).

Consider next the PDF (1.2). We know that as $A \to -\infty$ this reduces to (1.11), which is the matrix ensemble $\mathrm{LUE}_n|_{a=0}$. It is well known (see e.g. [16]) that the eigenvalue PDF for the LUE_n is realized by matrices of the form $X^\dagger X$ where X is an $n \times n$ matrix with i.i.d. complex Gaussian entries. Let us replace each complex element $x + iy$ of X by its 2×2 real matrix representation (4.18) and denote the corresponding $2n \times 2n$ matrix by $\bar{X}$. The eigenvalues of $\bar{X}$ are the eigenvalues of X except that in $\bar{X}$ each has multiplicity 2.

Theorem 4. *Let $\bar{X}$ be the $2n \times 2n$ real matrix constructed from the $n \times n$ complex Gaussian matrix as specified above, and let $\vec{x}$ denote a $2n \times 1$ real Gaussian vector, where in the Gaussians each independent part has variance $1/2$. The Gaussian random matrix*

$$M = \bar{X}^T \bar{X} + b\vec{x}\vec{x}^T$$

has eigenvalue PDF (1.2) with $A = 1 - 2/b$.

Proof. Arguing as in the proof of Theorem 3 we see that the eigenvalues of M consist of the eigenvalues of $\bar{X}^T \bar{X}$ with multiplicity 1, $y_1, \dots, y_n$ say, as well as n additional eigenvalues $x_1, \dots, x_n$ which must satisfy the interlacing condition (5.8). Making use of the result noted below (5.5) with $m_i = 2$, $\beta = 1$, it follows from (5.3) that the conditional distribution of the x's given the y's is proportional to

$$e^{-\sum_{i=1}^n (x_i - y_i)/b} \prod_{1 \le i < j \le n} \frac{(x_i - x_j)}{(y_i - y_j)}.$$

Relabelling the x's in (1.8) by $y's$, and forming the product with the conditional distribution shows that the eigenvalue PDF of M is proportional to

$$e^{-\sum_{i=1}^n (y_i + (x_i - y_i)/b)} \prod_{1 \le i < j \le n} (x_i - x_j)(y_i - y_j) \prod_{i,j=1}^n |x_i - y_j|,$$

and relabelling this coincides with (1.2) with $A = 1 - 2/b$ therein. □

Analogous remarks made after Theorem 3 also apply to Theorem 6. Thus by construction

$$\mathrm{even}(M) = \mathrm{LUE}_n|_{a=0},$$

and because when $b = 2$ the random matrix M has the same eigenvalue PDF as $\mathrm{LOE}_n|_{a=0} \cup \mathrm{LOE}_n|_{a=0}$ matrices, we thus have a matrix theoretic understanding of the relation [18]

$$\mathrm{even}(\mathrm{LOE}_n|_{a=0} \cup \mathrm{LOE}_n|_{a=0}) = \mathrm{LUE}_n|_{a=0}.$$

Also, the equation (5.10) holds with $\bar{X}$ replacing X. Here the matrix $\tilde{X}$ in the case $b = 1$ is equivalent to the $(2n+2) \times 2n$ version of $\bar{X}$ with the last row removed. Now before removing the row, such Gaussian matrices multiplied by their transpose have a doubly degenerate zero eigenvalue and n doubly degenerate eigenvalues with PDF $\mathrm{LUE}_n|_{a=1}$. Arguing as in the derivation of (5.11) we therefore conclude

$$\mathrm{odd}(M|_{b=1}) = \mathrm{LUE}_n|_{a=1}, \tag{5.12}$$

which like (5.11) can be verified by direct integration of the PDF (1.2) with $A = -1$ [19].

5.2 Laguerre limit of the three term recurrences

The joint PDF (3.24) has a well defined Laguerre limit, specified by changing variables $x_i \mapsto x_i/L$, $y_i \mapsto y_i/L$, setting $\beta = L/b, \beta_1 = L/b_1$ and taking the limit $L \to \infty$. This gives

$$\begin{aligned} L_{\rm o}^{(n,n-1)}(x,y) \quad := \quad & \frac{1}{\Gamma(1+\alpha)(\Gamma(k))^{n-1}} \frac{1}{\widetilde{W}_{n-1}(\alpha+\alpha_1+k,k;bb_1/(b+b_1))} \prod_{i=1}^{n} x_i^{\alpha} e^{-x_i/b} \prod_{1\le i<j\le n} |x_j - x_i| \\ & \times \prod_{i=1}^{n-1} y_i^{\alpha_1} e^{-y_i/b_1} \prod_{1\le i<j\le n} |y_j - y_i| \prod_{i=1}^{n} \prod_{j=1}^{n-1} |x_j - y_i|^{k-1} \end{aligned} \tag{5.13}$$

where

$$\begin{aligned} \widetilde{W}_n(a,k;b) &= \int_0^\infty dx_1 \cdots \int_0^\infty dx_n \prod_{l=1}^{n} x_l^a e^{-x_l/b} \prod_{1\le j<k\le n} |x_k - x_j|^k \\ &= b^{n(a+1+(n-1)k)} \prod_{j=1}^{n} \frac{\Gamma(1+kj)\Gamma(1+a+k(j-1))}{\Gamma(1+k)} \end{aligned}$$

and the x's and y's are interlaced according to

$$x_1 > y_1 > x_2 > y_2 > \cdots > y_{n-1} > x_n > 0. \tag{5.14}$$

Also of interest is the Laguerre limit of the marginal distribution (3.26),

$$\int_R dy_1 \cdots dy_{n-1}\, L_{\rm o}^{(n,n-1)}(x,y)\Big|_{\substack{\alpha_1=0 \\ 1/b_1=0}} = \frac{1}{\widetilde{W}_n(\alpha,k;b)} \prod_{i=1}^{n} x_i^{\alpha} e^{-x_i/b} \prod_{1\le j<k\le n} |x_k - x_j|^k =: L_{\rm o}^{(n,\text{-})}(x) \tag{5.15}$$

where R refers to the region (5.14).

Sampling from $L_{\rm o}^{(n,n-1)}(x,y)$ and $L_{\rm o}^{(n,\text{-})}(x)$ can be undertaken by taking the Laguerre limit of the three term recurrences in Section 4.2. Consider first the recurrence (4.38) determining the polynomial with zeros realizing the PDF $J_{\rm o}^{(n,\text{-})}(x)$. The Laguerre limit is obtained by scaling $x \mapsto x/L$, $w_1^{(j)} \mapsto v_1^{(j)}/L$, $w_2^{(j)} \mapsto v_2^{(j)}/L$, $w_0^{(j)} = 1$, where the $v_1^{(j)}, v_2^{(j)}$ are distributed according to the gamma distributions $\Gamma[b;(j-1)k]$, $\Gamma[b;(n-j)k+\alpha_0+1]$ respectively (here the notation $\Gamma[\sigma;s]$ refers to the density function proportional to $x^{s-1}e^{-s/\sigma}$). With $v_1^{(j)}, v_2^{(j)}$ so specified, and introducing the further scaling $A_j^{\#}(x) = L^{-j} B_j^{\#}(x)$ we see that the Laguerre limit of (4.38) reads

$$B_j^{\#}(x) = (x - v_2^{(j)}) B_{j-1}^{\#}(x) - x v_1^{(j)} B_{j-2}^{\#}(x). \tag{5.16}$$

This recurrence is to be solved subject to the initial conditions $B_{-1}^{\#}(x) = 0$ and $B_0^{\#}(x) = 1$. The zeros of $B_n^{\#}(x)$ are then distributed according to $L_{\rm o}^{(n,\text{-})}(x)|_{\alpha=\alpha_0}$.

To sample from $L_{\rm o}^{(n,n-1)}(x,y)$ we take an appropriate Laguerre limit of the procedure to sample from $J_{\rm o}^{(n,n-1)}(x,y)$ detailed below (4.38). Thus we use (5.16) to first compute $\{\tilde{B}_j^{\#}(x)\}_{j=0,\dots,n-1}$ where $\tilde{B}_j^{\#}(x)$ refers to $B_j^{\#}(x)$ with parameters $\alpha_0 \mapsto \alpha_0 + \alpha_1$, $1/b \mapsto 1/b + 1/b_1$. We then form the random polynomial

$$B_n(x) = (x - v_2^{(n)}) B_{n-1}^{\#}(x) - x v_1^{(n)} B_{n-2}^{\#}(x) \tag{5.17}$$

with $(v_1^{(n)}, v_2^{(n)})$ distributed according to $(\Gamma[1/b;(n-1)k], \Gamma[1/b;\alpha_0+1])$. The zeros of $(B_n(x), B_{n-1}^{\#}(y))$ then have the joint PDF $L_{\rm o}^{(n,n-1)}(x,y)$.

A recurrence to sample from $L_{\rm o}^{(n,\text{-})}(x)|_{\substack{\alpha=a-k(n-1)-1 \\ b=2}}$ has been given by Dumitriu and Edelman [12], as a corollary of their construction of a random tridiagonal matrix with this eigenvalue PDF. Denote by χ_p^2 the gamma distribution $\Gamma[2; p/2]$, and let χ_p denote the square root of a random variable with distribution χ_p^2. It was shown in [12] that the symmetric $n \times n$ random tridiagonal matrix

$$\begin{bmatrix} a_n & b_{n-1} & & & & \\ b_{n-1} & a_{n-1} & b_{n-2} & & & \\ & b_{n-2} & a_{n-2} & b_{n-3} & & \\ & \ddots & \ddots & \ddots & & \\ & & b_2 & a_2 & b_1 \\ & & & b_1 & a_1 \end{bmatrix}, \tag{5.18}$$

with the distribution of the elements specified by

$$a_n \sim \chi^2_{2a}, \quad a_i \sim \chi^2_{ki} + \chi^2_{2a-k(n-i)}, \quad b_i \sim \chi_{ki}\chi_{2a-k(n-i-1)} \ (i = n-1, \dots, 1)$$

has eigenvalue PDF given by $L_{\rm o}^{(n,\text{-})}(x)\big|_{\substack{\alpha=a-k(n-1)-1\\ b=2}}$. The random recurrence now follows from the fact that in general the characteristic polynomial of the bottom right $k \times k$ submatrix of (5.18) satisfies the three term recurrence

$$P_j(x) = (x - a_j)P_{j-1}(x) - b_{j-1}^2 P_{j-2}(x), \tag{5.19}$$

subject to the initial conditions $P_{-1}(x) = 0$, $P_0(x) = 1$. Note that (5.19) differs from (5.16).

The distribution $J_{\rm e}^{(n,n)}(x, y)$ as given by (3.29) also has a well defined Laguerre limit, obtained by writing $x_i \mapsto (1 - x_{n+1-i}/L)$, $y_i \mapsto (1 - y_{n+1-i}/L)$, setting $\alpha = L/b$, $\alpha_1 = L/b_1$, $\beta_1 = \alpha_1$ and taking $L \to \infty$. This gives

$$\begin{aligned} L_{\rm e}^{(n,n)}(x, y) = & \frac{1}{\Gamma(1+\alpha)(\Gamma(k))^n} \frac{1}{\widetilde{W}_n(\alpha_1, k, (b+b_1)/bb_1)} \\ & \times \prod_{i=1}^n e^{-x_i/b} y_i^{\alpha_1} e^{-y_i/b_1} \prod_{1\le i<j\le n} |x_j - x_i||y_j - y_i| \prod_{i,j=1}^n |x_j - y_i|^{k-1} \end{aligned} \tag{5.20}$$

where the x's and y's are interlaced according to

$$x_1 > y_1 > x_2 > y_2 > \cdots > x_n > y_n > 0. \tag{5.21}$$

We make note too of the Laguerre limit of (3.31),

$$\int_R dy_1 \cdots dy_n \, L_{\rm e}^{(n,n)}(x, y)\Big|_{1/b_1=0} = \frac{1}{\widetilde{W}_n(\alpha_1 + k, k, b)} \prod_{i=1}^n x_i^{\alpha_1+k} e^{-x_i/b} \prod_{1\le i<j\le n} |x_i - x_j|^{2k} =: L_{\rm e}^{(n,\text{-})}(x) \tag{5.22}$$

where R denotes the region (5.21).

To sample from $L_{\rm e}^{(n,n)}(x, y)$, we note that the Laguerre limit of $\{A_j^{\#{\rm e}}(x)\}_{j=0,\dots,n}$, used in (4.41) to sample from $J_{\rm e}^{(n,n)}(x, y)$, is given by $\{B_j^{\#{\rm e}}(x)\}_{j=0,\dots,n}$ where the $B_j^{\#{\rm e}}(x)$ are specified by the recurrence (5.16) but with $1/b \mapsto 1/b + 1/b_1$, $\alpha_0 \mapsto \alpha_1$ in the specification of the distribution of $(v_1^{(j)}, v_2^{(j)})$. Then taking the Laguerre limit of (4.41) tells us if we define

$$U_n(x) = u_2 B_n^{\#{\rm e}}(x) + u_1 B_{n-1}^{\#{\rm e}}(x), \qquad u_1 + u_2 = 1 \tag{5.23}$$

where u_1 is distributed according to $\Gamma[b; nk]$, then the joint distribution of the zeros $\{U_n(x), B_n^{\#{\rm e}}(y)\}$ is given by $L_{\rm e}^{(n,n)}(x, y)$. We remark that according to (5.22) the marginal distribution of $U_n(x)$ in the case $1/b_1 = 0$ is given by $L_{\rm e}^{(n,\text{-})}(x, y)$. But $B_n^{\#{\rm e}}(x)$ with $1/b_1 = 0$, $\alpha_0 \mapsto \alpha_1$ is the same as $B_n^{\#}(x)$ with $\alpha_0 \mapsto \alpha_1$ and thus has the PDF for its zeros given by (5.15) with $\alpha = \alpha_1$, and this in turn is identical to $L_{\rm e}^{(n,\text{-})}(x)|_{\alpha_1 \mapsto \alpha_1 - k}$. Thus we have the Laguerre limit of (4.44),

$$B_n^{\#}(x) = \Big(u_2 B_n^{\#}(x) + u_1 x B_{n-1}^{\#}(x)\Big)\Big|_{\alpha_1 \mapsto \alpha_1 - k}.$$

6 Gaussian interpolating ensembles

The random rational functions (4.7) and (5.2) have a counterpart

$$R^{\mathrm{G}}(\lambda) := \lambda + \sum_{j=1}^{n} \frac{w_j}{a_j - \lambda}, \tag{6.1}$$

where w_j is distributed according to $\Gamma[1; s_j]$, which is closely related to the Gaussian ensembles. Thus implicit in the work of Evans [14] on Gaussian analogues of the Selberg integral according to the method of Anderson is the following result for the PDF of the zeros of $R^G(\lambda)$.

Proposition 8. *Consider the random rational function (6.1). This has $n+1$ zeros $\lambda_1, \dots, \lambda_{n+1}$ restricted by the interlacing condition*

$$\lambda_1 > a_1 > \lambda_2 > a_2 > \cdots > a_n > \lambda_{n+1} \tag{6.2}$$

and the further requirement that

$$\sum_{l=1}^{n+1} \lambda_l = \sum_{l=1}^{n} a_l. \tag{6.3}$$

Subject to (6.2) and (6.3) the PDF of the zeros of (6.1) is given by

$$\frac{1}{\Gamma(s_1) \cdots \Gamma(s_n)} \frac{\prod_{1 \le j < k \le n+1} (\lambda_j - \lambda_k)}{\prod_{1 \le j < k \le n} (a_j - a_k)^{s_j + s_k - 1}} \prod_{j=1}^{n+1} \prod_{p=1}^{n} |\lambda_j - a_p|^{s_p - 1} \exp\Big(-\frac{1}{2}\Big(\sum_{j=1}^{n+1} \lambda_j^2 - \sum_{j=1}^{n} a_j^2 \Big)\Big). \tag{6.4}$$

Also of interest is the random rational function

$$\tilde{R}^{\mathrm{G}}(\lambda) := \lambda - w_0 + \sum_{j=1}^{n} \frac{w_j}{a_j - \lambda}, \tag{6.5}$$

where the w_j $(j = 1, \dots, n)$ are distributed as in (6.1) while w_0 is distributed according to N[0,1]. The PDF for the zeros of (6.5) is readily deduced from Proposition 8.

Corollary 4. *The zeros of the random rational function (6.5) have PDF (6.4) multiplied by $1/\sqrt{2\pi}$, except that the condition (6.3) is no longer required.*

Proof. The rational function $\tilde{R}^{\mathrm{G}}(\lambda)$ results from $\tilde{R}(\lambda)$ by making the replacements $\lambda \mapsto \lambda - w_0$, $a_j \mapsto a_j - w_0$. We then have that

$$\exp\Big(-\frac{1}{2}\Big(\sum_{j=1}^{n+1} \lambda_j^2 - \sum_{j=1}^{n} a_j^2\Big)\Big)\delta\Big(\sum_{j=1}^{n+1} \lambda_j - \sum_{j=1}^{n} a_j\Big) \mapsto e^{w_0^2/2} \exp\Big(-\frac{1}{2}\Big(\sum_{j=1}^{n+1} \lambda_j^2 - \sum_{j=1}^{n} a_j^2\Big)\Big)\delta\Big(\sum_{j=1}^{n+1} \lambda_j - \sum_{j=1}^{n} a_j - w_0\Big).$$

Multiplying this by $\frac{1}{\sqrt{2\pi}} e^{-w_0^2/2}$ (the distribution of w_0) and integrating over w_0 eliminates the delta function (and thus the restriction (6.3)) but leaves all other terms unchanged. $\square$

We remark that the random rational function (6.5) can be derived as a limit of the random rational function (4.21) with $n \mapsto n+1$ and $(w_0, \dots, w_n; w_{n+1})$ distributed according to $D_{n+2}[\alpha/2, s_1, \dots, s_n; \alpha/2]$. Thus if we write $\alpha = L^2$ and take $L \to \infty$ then the marginal distribution of w_0 and w_{n+1} have the asymptotic form $\frac{1}{2} + \frac{1}{2L}\mathrm{N}[0,1]$ while the w_i $(i = 1, \dots, n)$ have to leading order the marginal distribution $\frac{1}{L^2}\Gamma[1; s_i]$. It then follows from (4.21) with $x \mapsto \frac{1}{2}(1 - \frac{\lambda}{L})$ and $y_i \mapsto \frac{1}{2}(1 - \frac{a_i}{L})$ that we have

$$\frac{L}{2} \tilde{R}_{n+2}(x) \underset{L \to \infty}{\sim} \tilde{R}_n^{\mathrm{G}}(\lambda). \tag{6.6}$$

The random rational functions (6.1) and (6.5) occur in two closely related eigenvalue problems (see e.g. [8]). Thus let A be a real symmetric (complex Hermitian) matrix with eigenvalues $a_1 > a_2 > \cdots > a_n$ of multiplicities $m_1, \dots, m_n$. From A form a random matrix M of one extra column and one extra row by bordering A by a constant $\sqrt{b}$ times a vector of independent real standard Gaussians (complex Gaussians) as the final column, and the Hermitian conjugate of this as the final row (therefore in both the real ($\beta = 1$) and complex case ($\beta = 2$) we require the final entry of the vector to be real; let it have distribution $\mathrm{N}[0, \sqrt{2/\beta}]$). Thus if the final column of A is number n^*, then

$$[M]_{i,j} = A, \; [M]_{i,n^*+1} = [M^*]_{n^*+1,i} = \sqrt{b}[\vec{x}]_i, \; (1 \le i,j \le n^*) \quad [M]_{n^*+1,n^*+1} \sim N[0, \sqrt{2b/\beta}], \tag{6.7}$$

where here the symbol $\sim$ denotes 'has distribution'. A straight forward calculation shows that M has eigenvalues a_i with multiplicities $m_i - 1$, and $n+1$ further eigenvalues given by the zeros of the rational function (6.5) with w_0 (w_j) distributed according to $\mathrm{N}[0, \sqrt{2b/\beta}]$ ($\Gamma[2b/\beta, \beta m_j/2]$). It follows by scaling (6.5) that if we choose $c = 2b$ then the eigenvalue PDF of M is given by (6.4) with $\lambda_j \mapsto \sqrt{\beta/2b}\lambda_j$, $a_j \mapsto \sqrt{\beta/2b}a_j$.

If in the prescription (6.7) we choose $[M]_{n^*+1,n^*+1} = 0$ we find that M has eigenvalues a_i with multiplicities $m_i - 1$, and $n+1$ further eigenvalues given by the zeros of the random rational function (6.1) with w_i as specified in the above paragraph. Note that the condition (6.3) then has the interpretation as the statement that $\mathrm{Tr}(A) = \mathrm{Tr}(M)$.

6.1 Construction of Gaussian interpolating matrix ensembles

Following a strategy analogous to that used in the construction of random matrices realizing the Jacobi and Laguerre interpolating ensembles, we can use the eigenvalue problem relating to (6.5) to construct random matrices with eigenvalue PDFs realizing certain Gaussian interpolating ensembles. In particular we can construct random matrices with eigenvalue PDF of the form

$$\frac{1}{C} \prod_{i=1}^{n+1} e^{-c_1 x_i^2/2} \prod_{1 \le i < j \le n+1} (x_i - x_j) \prod_{i=1}^{n} e^{-c_2 y_i^2/2} \prod_{1 \le i < j \le n} (y_i - y_j), \tag{6.8}$$

where

$$x_1 > y_1 > \cdots > y_n > x_{n+1}, \tag{6.9}$$

which with $c_2 = 0$ reduces to (2.48), and the eigenvalue PDF

$$\frac{1}{C} \prod_{i=1}^{n+1} e^{-c_1 x_{2i-1}^2/2} \prod_{i=1}^{n} e^{-c_2 x_{2i}^2/2} \prod_{1 \le i < j \le 2n+1} (x_i - x_j) \tag{6.10}$$

where

$$x_1 > x_2 > \cdots > x_{2n+1}. \tag{6.11}$$

To obtain (6.8) we choose the matrix A in (6.7) to be an $n \times n$ member of the GUE (see e.g. [16] for the precise definition of such matrices), and we extend A so specified to a $2n \times 2n$ real matrix by replacing each complex element by its 2×2 real matrix representation (4.18). Following the strategies of the proofs of Theorems 2 and 4 it follows that M as specified by (6.7) with $n^* = 2n$, $\beta = 1$ has the n eigenvalues of A with multiplicity 1, $y_1, \dots, y_n$ say, and a further $n+1$ eigenvalues $x_1, \dots, x_{n+1}$ say interlaced according to (6.9). Furthermore it follows that the joint eigenvalue PDF is given by (6.8) with

$$c_1 = \frac{1}{2b}, \qquad c_2 = -\frac{1}{2b} + 2. \tag{6.12}$$

The above construction gives

$$\mathrm{even}(M) = \mathrm{GUE}_n. \tag{6.13}$$

Now with $b = 1/2$ we have $c_1 = c_2 = 1$ and we recognize (6.8) as the eigenvalue PDF for $\mathrm{GOE}_{n+1} \cup \mathrm{GOE}_n$ (see e.g. [18]). Thus we have a matrix theoretic understanding of the identity [18]

$$\mathrm{even}(\mathrm{GOE}_{n+1} \cup \mathrm{GOE}_n) = \mathrm{GUE}_n. \tag{6.14}$$

We note too that with $b = 1/4$ the matrix M coincides with the upper left $(2n+1) \times (2n+1)$ block of the real matrix representation of a $(n+1) \times (n+1)$ GUE matrix. By an argument analogous to the derivation of (5.11) we must therefore have

$$\mathrm{odd}(M|_{b=1/4}) = \mathrm{GUE}_{n+1}. \tag{6.15}$$

As $b = 1/4$ corresponds to $c_2 = 0$, this identity is relevant to (2.48).

For the realization of (6.10) we choose the matrix A in (6.7) to be a $n \times n$ member of the GSE (by definition — see e.g. [19] — the elements of such matrices are real quaternions, so as a complex matrix A is $2n \times 2n$). The eigenvalues are doubly degenerate, with the independent eigenvalues $y_1, \dots, y_n$ say having distribution

$$\frac{1}{C} \prod_{l=1}^{n} e^{-y_l^2} \prod_{1 \le j < k \le n} (y_j - y_k)^4.$$

Here we follow the strategy of the proofs of Theorems 1 and 3 to conclude that (6.7) with this choice of A and $n^* = 2n$, $\beta = 2$ has the n eigenvalues of A and with multiplicity 1, $x_2, x_4, \dots, x_{2n}$ say, and a further $n+1$ eigenvalues $x_1, x_3, \dots, x_{2n+1}$ say, interlaced according to (6.11) and with eigenvalue PDF (6.10) with

$$c_1 = \frac{1}{b}, \qquad c_2 = -\frac{1}{b} + 2.$$

Since by construction

$$\mathrm{even}(M) = \mathrm{GSE}_n,$$

and with $b = 1$ and thus $c_1 = c_2 = 1$ the PDF (6.10) reduces to the PDF for GOE_{2n+1}, we thus have a matrix theoretic understanding of the relation [19]

$$\mathrm{even}(\mathrm{GOE}_{2n+1}) = \mathrm{GSE}_n.$$

Furthermore, with $b = 1/2$ the matrix M coincides with the upper left $(2n+1) \times (2n+1)$ block of the complex representation of a $(n+1) \times (n+1)$ GSE matrix, and so we must have

$$\mathrm{odd}(M|_{b=1/2}) = \mathrm{GSE}_{n+1}.$$

6.2 Gaussian limit of the three term recurrences

The PDFs (6.8) and (6.10) are special cases of a limiting form of the joint PDF (3.24). Thus in (3.24) let us change variables $x_i \mapsto (\frac{1}{2} - \frac{x_i}{2L})$, $y_i \mapsto (\frac{1}{2} - \frac{y_i}{2L})$, set $\alpha = \beta = aL^2$, $\alpha_1 = \beta_1 = a_1 L^2$ and take $L \to \infty$. We then obtain the joint PDF

$$G_{\mathrm{o}}^{(n,n-1)}(x,y) := \Big(\frac{a}{\pi}\Big)^{1/2} \frac{(2a)^{(n-1)k}}{(\Gamma(k))^{n-1}} \frac{1}{M_{n-1}(k; 2(a+a_1))}$$
$$\times \prod_{i=1}^{n} e^{-a x_i^2} \prod_{1 \le i < j \le n} |x_j - x_i| \prod_{i=1}^{n-1} e^{-a_1 y_i^2} \prod_{1 \le i < j \le n-1} |y_j - y_i| \prod_{i=1}^{n} \prod_{j=1}^{n-1} |x_j - y_i|^{k-1} \tag{6.16}$$

where

$$M_n(k;c) \;:=\; \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_n \, e^{-(c/2)\sum_{l=1}^n x_l^2} \prod_{1\le i<j\le n} |x_j - x_i|^{2k}$$
$$= c^{-n/2-kn(n-1)/2}(2\pi)^{n/2} \prod_{j=0}^{n-1} \frac{\Gamma(1+(j+1)k)}{\Gamma(1+k)}$$

and the x's and y's are interlaced according to

$$\infty > x_1 > y_1 > x_2 > y_2 > \cdots > y_{n-1} > x_n > -\infty. \tag{6.17}$$

We make note of the special marginal distribution

$$\int_R dy_1 \cdots dy_{n-1}\, G_{\rm o}^{(n,n-1)}(x,y)\Big|_{a_1=0} = \frac{1}{M_n(k;2a)} \prod_{i=1}^n e^{-2ax_i^2} \prod_{1\le i<j\le n} |x_i - x_j|^{2k} =: G_{\rm o}^{(n,\text{-})}(x), \tag{6.18}$$

which is equivalent to the integration formulas (3.26) and (5.15).

To sample from $G_{\rm o}^{(n,\text{-})}(x)$ we can take the Gaussian limit of the three term recurrence (4.38). First we note that with $(w_0^{(j)}, w_1^{(j)}, w_2^{(j)})$ distributed as specified below (4.38), setting $\alpha_0 = \beta_0 = aL^2$ and taking $L \to \infty$, the marginal distributions of $w_0^{(j)}$ and $w_2^{(j)}$ have the asymptotic form $\frac{1}{2} + \frac{1}{2L}\mathrm{N}[0, \frac{1}{\sqrt{2a}}]$ while $w_1^{(j)}$ has the leading order marginal distribution $\frac{1}{L^2}\Gamma[\frac{1}{2a}; (j-1)k]$ (c.f. the statements above (6.6)). Thus by also writing $x \mapsto \frac{1}{2}(1 - \frac{x}{L})$, $A_j^{\#}(x) \mapsto (-2L)^{-j} C_j^{\#}(x)$, we see that in the Gaussian limit (4.38) reduces to

$$C_j^{\#}(x) = (x-r)C_{j-1}^{\#}(x) - s^{(j-1)} C_{j-2}^{\#}(x) \tag{6.19}$$

where r has distribution $\mathrm{N}[0, \frac{1}{\sqrt{2a}}]$ while $s^{(j-1)}$ has distribution $\Gamma[\frac{1}{2a}; (j-1)k]$. With the initial conditions $C_{-1}^{\#}(x) = 0$, $C_0^{\#}(x) = 1$, we have that the zeros of $C_n^{\#}(x)$ have PDF $G_{\rm o}^{(n,\text{-})}(x)$. The recurrence (6.19) has the structure (5.19) and thus can be viewed as specifying the characteristic polynomial for a corresponding random tridiagonal matrix (5.18). In fact this is precisely the random tridiagonal matrix found by Dumitriu and Edelman [12] and shown to have eigenvalue PDF given by $G_{\rm o}^{(n,\text{-})}(x)$.

To sample from $G_{\rm o}^{(n,n-1)}(x,y)$, the Gaussian limit of the procedure to sample from $J_{\rm o}^{(n,n-1)}(x,y)$ detailed below (4.38). Thus we use (6.19) to generate $\{\tilde{C}_j^{\#}(x)\}_{j=0,\dots,n-1}$ where $\tilde{C}_j^{\#}(x)$ refers to $C_j^{\#}(x)$ with parameter $a \mapsto a + a_1$. We then form the random polynomial

$$C_n(x) = (x-r)\tilde{C}_{n-1}^{\#}(x) - s^{(n-1)}\tilde{C}_{n-2}^{\#}(x)$$

with $(r, s^{(n-1)})$ distributed according to $(\mathrm{N}[0, \frac{1}{\sqrt{2a}}], \Gamma[\frac{1}{2a}; (n-1)k])$. The PDF $G_{\rm o}^{(n,n-1)}(x,y)$ is then realized by the zeros of $(C_n(x), \tilde{C}_{n-1}^{\#}(y))$. Equivalently we can realize the PDF in terms of the eigenvalues of a random $n \times n$ tridiagonal matrix and its lower right $(n-1)\times(n-1)$ submatrix.

Theorem 5. *Consider the symmetric tridiagonal matrix (5.18). Let the elements be random with distributions*

$$a_i \sim \mathrm{N}[0, \frac{1}{\sqrt{2(a+a_1)}}], \quad b_{i-1}^2 \sim \Gamma[\frac{1}{2(a+a_1)}; (i-1)k] \quad (i = 1, \dots, n-1)$$
$$a_n \sim \mathrm{N}[0, \frac{1}{\sqrt{2a}}], \quad b_{n-1}^2 \sim \Gamma[\frac{1}{2a}; (n-1)k].$$

The joint distribution of the eigenvalues $x_1, \dots, x_n$ of this matrix, and the eigenvalues $y_1, \dots, y_{n-1}$ of the $(n-1)\times(n-1)$ bottom right submatrix, is given by $G_{\rm o}^{(n,n-1)}(x,y)$.

Appendix A

Geometrical RSK

The Robinson-Schensted-Knuth correspondence between non-negative integer matrices and pairs of semi-standard tableaux of the same shape has a geometrical representation relating the matrix to paths defining the interface of a sequence of growth models [24] (for closely related geometrical representations see [20] and [15]). Here will will show that coordinates specifying the paths can be used to deduce the equation (2.10) and also the joint probability (2.4).

First we revise the construction of [24], wherein each distinct $n \times n$ non-negative square matrix $X = [x_{i,j}]_{i,j=1,\dots,n}$ (rows counted from the bottom), with entries $x_{i,j}$ weighted $(1 - a_i b_j)(a_i b_j)^{x_{i,j}}$, is put into a one-to-one correspondence with a set of at most n non-intersecting weighted lattice paths, starting at $(x, y) = (-(2n - 1/2), l - 1)$ and finishing at $(x, y) = ((2n - 1/2), l - 1)$ $(l = 1, 2, \dots)$. The lth member of the set — the path starting and finishing along $y = -(l - 1)$ — will be referred to as the level-l path. Each level-l path can be regarded as a pair of paths because the weights and the allowed steps are different depending on $x < 0$ or $x > 0$. Thus the first (second) member of the pair starts at $x = -(2n - 1/2)$ $(x = (2n - 1/2))$ and goes either right (left) in steps of two units, or up in integer amounts at $x = -(2n + 3/2 - 2j)$ $(x = (2n + 3/2 - 2j)$ with each unit regarded as a step weighted by b_j (a_j), until it reaches $x = -1/2$ $(x = 1/2)$ where both paths must have the same final y-coordinate (see Figure 1).

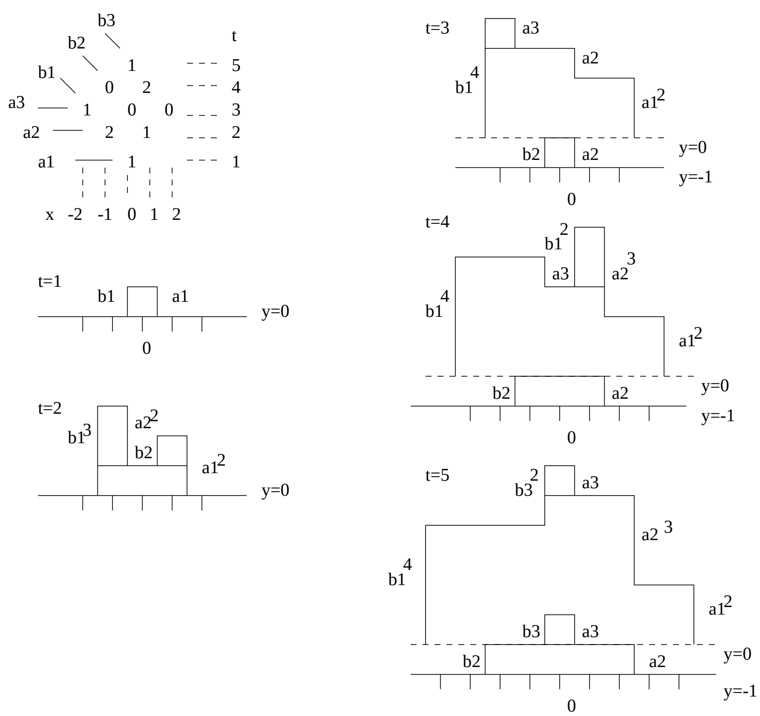

Figure 1: The RSK mapping from a weighted integer matrix to a set of weighted non-intersecting paths. The mapping is invertible and so is a bijection.

Let μ_l denote the maximum height of the level-l path, which is the displacement of this path at $x = \pm 1/2$. Because the paths cannot intersect we must have $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ so $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ forms a partition. It is a standard result (see e.g. [34]) that the total weight of all non-intersecting paths of the specified type for $x < 0$, initially equally spaced at $y = 0, \dots, -(n-1)$ along $x = -(2n - 1/2)$ and finishing at $y = \mu_1, \mu_2 - 1, \dots, \mu_n - (n-1)$ along $x = -1/2$ is given by the Schur polynomial $s_\mu(b_1, \dots, b_n)$. Similarly the total weight of all non-intersecting paths of the specified type for $x > 0$, initially equally spaced at $y = 0, \dots, -(n-1)$ along $x = (2n - 1/2)$ and finishing at $y = \mu_1, \mu_2 - 1, \dots, \mu_n - (n-1)$ along $x = 1/2$ is given by the Schur polynomial $s_\mu(a_1, \dots, a_n)$. Furthermore it is another standard result (see e.g. [34]) that each set of non-intersecting paths from $x = -(2n - 1/2)$ to $x = -1/2$ is equivalent to a semi-standard tableau of shape μ, content n, as each set of paths from $x = (2n - 1/2)$ to $x = 1/2$. Thus for the non-intersecting lattice paths of Figure 1 we obtain the pair of tableaux

1	1	2	2	2	3
2	3				

1	1	1	1	3	3
2	3				

the first corresponding to the paths $x > 0$ and the second the paths $x < 0$. In the former (latter) each occurence of a_j (b_j) in the level-l path is recorded as a box labelled j in the lth row of the tableau.

Thus if we accept the mapping, the probability (2.3) is immediate (an unequal number of a and b weights can be achieved by simply setting some of them equal to zero and using the stability property of the Schur polynomial, $s_\mu(a_1, \dots, a_{n-1}, 0) = s_\mu(a_1, \dots, a_{n-1})$).

To derive (2.4) and (2.10) we must investigate the details of the mapping, which takes the form of a cascade of polynuclear growth models (see Figure 1). We rotate the non-negative matrix $X = [x_{i,j}]_{i,j=1,\dots,n}$ 45° anti-clockwise and label the horizontal rows of the rotated matrix by $t = 1, 2, \dots, 2n-1$ and the vertical columns by $x = 0, \pm 1, \dots, \pm(n-1)$ where $x = 0$ corresponds to the diagonal $i = j$ of X (recall that the rows are being counted from the bottom). The entries $x_{i,j}$ in the matrix for successive t values ($t = i + j - 1$) are heights of weighted 'nucleation events' — columns of unit width and height $x_{i,j}$ centred about the corresponding x-coordinate which are placed on top of the profile formed by earlier nucleation events and their growth and weighted by $(a_i b_j)^{x_{i,j}}$ (in addition the matrix has a normalization weighting of $\prod_{i,j=1}^n (1 - a_i b_j)$ independent of the entries). Thus at $t = 1$ there is a nucleation event at $x = 0$ which consists of a column of width 1, height x_{11} and weight $(a_1 b_1)^{x_{11}}$ marked on the line at $y = 0$ in the xy-plane. In general, as $t \mapsto t + 1$ the profile of all nucleation events so far recorded is to 'grow' one unit in the $-x$ direction and one unit in the $+x$ direction. Thus in going from $t = 1$ to $t = 2$ the nucleation event centred at $x = 0$ of height x_{11} now has width 3 units. On top of this profile, centred at $x = -1$ and $x = 1$ nucleation events of unit width and height x_{21}, x_{12} and weight $(a_2 b_1)^{x_{21}}$, $(a_1 b_2)^{x_{12}}$ respectively are then drawn. In now going from $t = 2$ to $t = 3$ this new profile is to grow one unit to the left and one unit to the right. In so doing we see that an overlap of width one unit and height $\min(x_{21}, x_{12})$ will occur. This overlap is ignored in the first diagram (profile on $y = 0$), and recorded instead as a profile on the line immediately below (here $y = -1$). The process is repeated with these rules until the nucleation event of height x_{nn}, weight $(a_n b_n)^{x_{nn}}$ at $t = 2n - 1$ has been recorded above $x = 0$ on the first diagram. In this way we obtain the sought mapping from a weighted non-negative integer matrix to weighted non-intersecting lattice paths. The mapping is easily seen to be invertible, and so is a bijection. With each path considered as a pair of paths depending on whether $x < 0$ or $x > 0$, and then the set of paths for $x < 0$ and the set of paths for $x > 0$ recorded as a pair of semi-standard tableaux, this gives the same correspondence between non-negative integer matrices and pairs of semi-standard tableaux of the same shape as the Robinson-Schensted-Knuth algorithm (this last point follows because, as noted in [24], the above algorithm can be viewed as a graphical presentation of the matrix-ball construction of Fulton [20], which has been shown to give the RSK correspondence).

Of crucial interest to us is the sequence of maximum displacements $\lambda_l(n_1, n_2)$ of the level-l path obtained by applying the growth process to the truncation X_{n_1,n_2} say of the matrix X to the first n_1 rows and n_2 columns, extended to a square matrix of dimension $\max(n_1,n_2)\times\max(n_1,n_2)$ by appending rows of zeros to the top (or columns of zeros to the right, as appropriate). Consider first the level-1 path. It follows from the rules of the growth process that $\lambda_1(n_1,n_2)$ results by adding x_{n_1,n_2} to the maximum of the height at $x=-1$ and the height at $x=1$ in the previous time step ($h_1(n_1,n_2-1)$ and $h_1(n_1-1,n_2)$ respectively say; see the change in the height at the origin in going from $t=4$ to $t=5$ in Figure 1). Thus

$$\lambda_1(n_1,n_2)=\max\Big(h_1(n_1,n_2-1),h_1(n_1-1,n_2)\Big)+x_{n_1,n_2}.$$

But again from the rules of the growth process

$$h_1(n_1,n_2-1)=\lambda_1(n_1,n_2-1),\qquad h_1(n_1-1,n_2)=\lambda_1(n_1-1,n_2)\tag{A.1}$$

since the nucleation events in column n_2, and rows $1,2,\dots,n_1-1$ from the bottom cannot contribute to $h_1(n_1,n_2-1)$, and similarly the nucleation events in row n_1 from the bottom and columns $1,2,\dots,n_2-1$ cannot contribute to $h_1(n_1-1,n_2)$ (see Figure 2). Therefore we obtain the recurrence

$$\lambda_1(n_1,n_2)=\max\Big(\lambda_1(n_1,n_2-1),\lambda_1(n_1-1,n_2)\Big)+x_{n_1,n_2}\tag{A.2}$$

which with the boundary condition

$$\lambda_1(0,j)=\lambda_1(i,0)=0$$

uniquely specifies $\{\lambda_1(i,j)\}_{i,j=1,\dots,n}$. One observes that $L(n_1,n_2)$ as specified by (2.2) satisfies the very same recurrence (A.2), and indeed one of the primary relations in the RSK correspondence is

$$\lambda_1(n_1,n_2)=L(n_1,n_2).\tag{A.3}$$

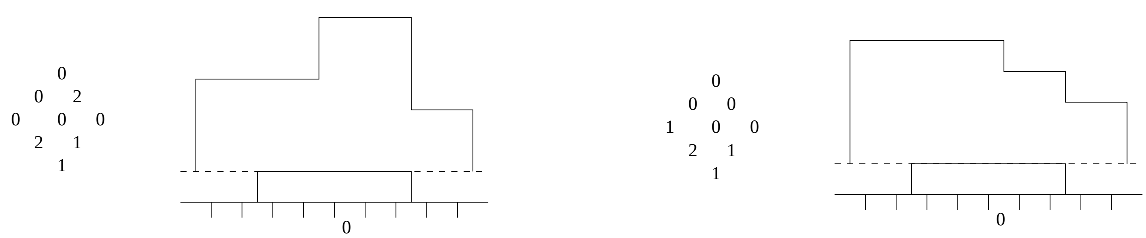

Figure 2: The path diagram for the matrix of Figure 1 with the final row (column) set equal to zero. This can be deduced from the $t=4$ diagram of Figure 1 with a_3 (b_3) set equal to zero. The maximum heights therefore coincide with the maximum height at $x=1/2$ ($x=-1/2$) in this diagram.

For the maximum displacements $\lambda_l(n_1,n_2)$ of the level-l path, $l>1$, we see from the derivation of (A.2) that

$$\lambda_l(n_1,n_2)=\max\Big(\lambda_l(n_1,n_2-1),\lambda_l(n_1-1,n_2)\Big)+x^{(l-1)}_{n_1,n_2}$$

where $x^{(l-1)}_{n_1,n_2}$ is the height of an overlap event (if any) which occurs in the growth of the nucleation events corresponding to x_{n-1,n_2-1} or x_{n_1-1,n_2}. We remark that $[x^{(l-1)}_{i,j}]_{i,j=1,\dots,n}$ defines the lth member of the sequence of matrices in Fulton's matrix ball construction [20]. The rules of the growth process give

$$\begin{aligned}x^{(l-1)}_{n_1,n_2}&=\min\Big(h_{l-1}(n_1,n_2-1),h_{l-1}(n_1-1,n_2)\Big)-h_{l-1}(n_1-1,n_2-1)\\&=\min\Big(\lambda_{l-1}(n_1,n_2-1),\lambda_{l-1}(n_1-1,n_2)\Big)-\lambda_{l-1}(n_1-1,n_2-1)\end{aligned}$$

(see the nucleation events created in level 2 in going from $t = 2$ to $t = 3$, and going from $t = 4$ to $t = 5$ in Figure 1) and so for $l > 1$

$$\begin{aligned} \lambda_l(n_1, n_2) &= \max\Big(\lambda_l(n_1, n_2 - 1), \lambda_l(n_1 - 1, n_2)\Big) \\ &\quad + \min\Big(\lambda_{l-1}(n_1, n_2 - 1), \lambda_{l-1}(n_1 - 1, n_2)\Big) - \lambda_{l-1}(n_1 - 1, n_2 - 1), \end{aligned} \tag{A.4}$$

which with the boundary condition

$$\lambda_l(0, j) = \lambda_l(i, 0) = 0 \tag{A.5}$$

and knowledge of $\{\lambda_1(i, j)\}_{i,j=1,\dots,n}$ from (A.2) uniquely specifies $\{\lambda_l(i, j)\}_{i,j=1,\dots,n}$. For recurrences closely related to (A.4), see [25, 30].

We have defined $\lambda_l(n_1, n_2)$ as the maximum height of the level-l path resulting from applying the growth process to the truncation X_{n_1,n_2} of the original matrix X. But there is another equally important interpretation of $\lambda_l(n_1, n_2)$. Thus consider n_1, n_2 as fixed and apply the growth process to X_{n_1,n_2}. Then we can see, arguing as in the justification of the equalities (A.1), that $\lambda_l(n_1, j)$ is equal to the displacement of the level-l path at $x = -2n^* - 3/2 + 2j$, while $\lambda_l(i, n^*)$ is equal to the displacement of the level-l path at $x = 2n^* + 3/2 - 2i$, where $n^* = \max(n_1, n_2)$. It follows immediately from this interpretation that (see Figure 3)

$$\begin{aligned} &\lambda_l(i, j) = 0 \qquad \text{for} \quad l > \max(n_1, n_2) \\ &\lambda_l(i, j) \geq \lambda_l(i, j - 1) \geq \lambda_{l+1}(i, j) \\ &\lambda_l(i, j) \geq \lambda_l(i - 1, j) \geq \lambda_{l+1}(i, j), \end{aligned} \tag{A.6}$$

and furthermore

$$\begin{aligned} \sum_{l=1}^{n} \Big(\lambda_l(n, j) - \lambda_l(n, j - 1)\Big) &= \sum_{i=1}^{n} x_{ij} \\ \sum_{l=1}^{n} \Big(\lambda_l(i, n) - \lambda_l(i - 1, n)\Big) &= \sum_{j=1}^{n} x_{ij}. \end{aligned} \tag{A.7}$$

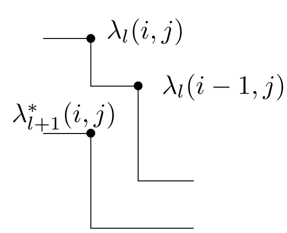

Figure 3: Graphical demonstration of the final inequality in (A.6), where here all marked displacements are with respect to the line $y = -(l - 1)$, and $\lambda^*_{l+1}(i, j) := \lambda_{l+1}(i, j) + 1$.

We are now in a position to derive the joint probability (2.4). Let X_{n_1,n_2+1} be an integer matrix mapping to a pair of semi-standard tableaux of shape μ under the RSK correspondence. Now each part μ_l of μ is equal to the maximum displacement μ_l of the level-l path, and so from the above discussion $\mu_l = \lambda_l(n_1, n_2{+}1)$. Similarly, with κ denoting the shape of the pair of tableaux resulting from applying the RSK correspondence to the truncation X_{n_1,n_2} of X_{n_1,n_2+1} obtained by deleting the rightmost column, the above discussion shows $\kappa_l = \lambda_l(n_1, n_2)$. The relations (A.6) immediately give (2.5) in the case $n_2 \geq n_1$, and (2.7) for $n_2 < n_1$. Furthermore, from the geometrical RSK mapping illustrated in Figure 1, the joint probability is seen to be equal to $\prod_{i=1}^{n_1}\prod_{j=1}^{n_2+1}(1-a_ib_j)$ (the normalization weighting), times the total weight of all non-intersecting paths from $(2n^*+1/2, l-1)$ to $(1/2, l-1)$ $(l = 1,\dots,n^*)$ with $n^* = \max(n_1, n_2+1)$, weighting a_j for each unit up step at $x = 2n-3/2+2j$, $(j = 1,\dots,n_1)$, times the total weight of all non-intersecting paths from $(-2n^*-1/2, l-1)$ to $(-3/2, l-1)$, weighting b_j for each unit up step at $x = 2n^*-3/2+2j$ $(j = 1,\dots,n_1)$, times the step weight b_{n_2+1} raised to the power of the difference in the maximum height of the level-l path for $x > 0$ (μ_l) and the level-l path for $x < 0$ (κ_l) summed over l. Writing the total weights of the paths in terms of Schur polynomials we see that (2.4) results for $n_2 \ge n_1$, and (2.4) modified by (2.7) results for $n_2 < n_1$.

To derive (2.10) we first note that in the special case $X = [x_{i,j}]_{i,j=1,\dots,n}$ is symmetric about $i = j$, we must have $\lambda_l(i,j) = \lambda_l(j,i)$. Hence it follows from (A.2) and (A.4) that then

$$\begin{aligned}\lambda_1(i,i) &= \lambda_1(i,i-1) + x_{i,i}\\ \lambda_l(i,i) &= \lambda_l(i,i-1) + \lambda_{l-1}(i,i-1) - \lambda_{l-1}(i-1,i-1), \quad l > 1.\end{aligned}$$

Forming appropriate linear combinations of these equations shows

$$\sum_{l=1}^{i}(-1)^{l-1}\lambda_l(i,i) - \sum_{l=1}^{i-1}(-1)^{l-1}\lambda_l(i-1,i-1) = x_{i,i},$$

and summing this equation over i from 1 to n gives (2.10).

Continuous RSK

In the above description of the RSK correspondence a weighted non-negative integer matrix $X = [x_{ij}]_{i,j=1,\dots,n}$ has been mapped bijectively to a set of non-intersecting weighted lattice paths. Because for given maximum displacements $\mu_1,\dots,\mu_n$ the total weight of the non-intersecting paths given by a product of Schur polynomials of the same index μ, this allows the probability that the integer matrix matrix maps to such lattice paths to be specified by (2.3).

Also of interest is the case when the matrix X consists of non-negative real valued random variables x_{ij} distributed according to the exponential distribution

$$\Pr(x_{ij} \in [y, y+dy]) = (\alpha_i + \beta_j)e^{-(\alpha_i+\beta_j)y}dy, \quad y \ge 0. \tag{A.8}$$

Using this distribution to define a probability measure on X, we see (as noted in [24]) that the RSK correspondence gives a bijective mapping to a set of non-intersecting paths with a certain probability measure. The description of the lattice paths differs in some details to the discrete case. First, their steps are continuous in the y-direction and discrete in the x-direction. All paths start along $y = 0$, with the level-l path starting at $(x,y) = (-(2n+3/2-2l), 0)$ and finishing at $(x,y) = (2n+3/2-2l, 0)$. Vertical steps can occur at $x = -(2n+3/2-2j)$ $(x = 2n+3/2-2j)$, $j = l,\dots,n$, with the constraints that the non-zero height of the level-l path is greater than that of the level-$(l+1)$ path, and the height of each level-l path is weakly increasing going from $x = -(2n-1/2)$ to $x = -1/2$ and weakly decreasing going from $x = 1/2$ to $x = 2n-1/2$.

We are interested in the probability density that the level-l paths will have maximum displacement y_l, $l = 1,\dots,n$. By the choice of the exponential distribution (A.8), the RSK correspondence shows that at $x = -(2n+3/2-2j)$ $(x = 2n+3/2-2j)$ the vertical increment of each level-l path with $l \le j$ is a random variable proportional to $e^{-\beta_j y}$ $(e^{-\alpha_j y})$ — the normalization $\prod_{i,j=1}^{n}(\alpha_i+\beta_j)$ is taken as an overall factor — which is conditioned so that the paths for levels $1, 2, \dots, j$ do not intersect, and furthermore at level-l the sum of the vertical increments for both $x > 0$ and $x < 0$ is equal to y_l.

The total weight of a single path with vertical increments of length v_j at $x = -(2n + 3/2 - 2j)$ $(j = l, \dots, n)$, weighted by $e^{-\beta_j v_j}$ and constrained so that $\sum_{j=l}^n v_j = y_k$ is given by

$$\int_0^\infty d\delta_l \, e^{-\beta_l v_l} \cdots \int_0^\infty d\delta_n \, e^{-\beta_n v_n} \, \delta\Big(y_k - \sum_{j=l}^n v_j\Big) = \sum_{j=l}^n \frac{e^{-\beta_j y_k}}{\prod_{\substack{\mu=l \\ \mu \neq j}}^n (\beta_j - \beta_\mu)} =: u_l(\{\beta_j\}_{j=l,\dots,n}; y_k).$$

The extension of the Karlin-McGregor theorem used in [24] gives that the corresponding total weight of the set of continuous non-intersecting paths for $x < 0$ is $\det[u_l(\{\beta_j\}_{j=l,\dots,n}; y_k)]_{k,l=1,\dots,n}$. Similarly for $x > 0$ the total weight of the set of continuous paths is $\det[u_l(\{\alpha_j\}_{j=l,\dots,n}; y_k)]_{k,l=1,\dots,n}$. Thus the sought probability density is

$$\prod_{i,j=1}^n (\alpha_i + \beta_j) \, \det[u_l(\{\alpha_j\}_{j=l,\dots,n}; y_k)]_{k,l=1,\dots,n} \det[u_l(\{\beta_j\}_{j=l,\dots,n}; y_k)]_{k,l=1,\dots,n}. \tag{A.9}$$

In the special case

$$\alpha_i = a + (i-1), \qquad \beta_j = \bar{a} + (j-1)c$$

we have the simplifications

$$u_l(\{\alpha_j\}_{j=l,\dots,n}; y_k) = \frac{e^{-(a+(l-1))y_k}}{(n-l)!}(1 - e^{-y_k})^{n-l}, \quad u_l(\{\beta_j\}_{j=l,\dots,n}; y_k) = \frac{e^{-(\bar{a}+(l-1)c)y_k}}{c^{n-l}(n-l)!}(1 - e^{-cy_k})^{n-l}$$

and, after making use of the Vandermonde determinant identity, we obtain the corresponding simplification of (A.9)

$$\prod_{j=1}^n \frac{\Gamma(a + \bar{a} + (j-1)c + n)}{\Gamma(a + \bar{a} + (j-1)c) c^{j-1} \Gamma^2(j)} e^{-(a+\bar{a})\sum_{j=1}^n y_j} \prod_{1 \le i < j \le n} (e^{-cy_j} - e^{-cy_i})(e^{-y_j} - e^{-y_i}). \tag{A.10}$$

Appendix B

The summation identity (3.17)

Substituting (3.13) in (3.17) shows that we have the normalization identity

$$\frac{1}{\varepsilon_{t^n,t}(P_\mu)} \sum_\kappa \varepsilon_{t^{n-1},t}(P_\kappa) \psi_{\mu/\kappa}(q,t) t^{|\kappa|} = 1 \tag{B.1}$$

where the summation over κ is over the region (3.14). Making use of (3.2) and (3.7) we find that with

$$t = q^k, \quad y_{n-i} := q^{\kappa_i} t^{n-1-i}, \quad x_{n+1-i} := q^{\mu_i} t^{n-i}, \tag{B.2}$$

(B.1) can be rewritten to read

$$\begin{aligned} \sum_y \prod_{j=1}^{n-1} y_j \prod_{1 \le i < j \le n-1} (y_i - y_j) \prod_{1 \le i \le j \le n-1} (qy_j/x_i; q)_{k-1} (qy_i/x_{j+1}; q)_{k-1} \\ = \frac{((q;q)_{k-1})^n}{(q;q)_{kn-1}} \prod_{1 \le i < j \le n} (x_i - x_j)(qx_i/x_j; q)_{k-1} (qx_j/x_i)_{k-1} \end{aligned} \tag{B.3}$$

where the summation over y is over the regions

$$y_i = x_i, qx_i, q^2 x_i, q^3 x_i, \dots, x_{i+1} \quad (i = 1, \dots, n-1). \tag{B.4}$$

This can be viewed as a multidimensional q-integral. It is the special case $s_1 = \cdots = s_n = k$ of Evans [13] q-integral

$$\begin{aligned}\sum_y \prod_{j=1}^{n-1} y_j \prod_{1\le i<j\le n-1} (y_i - y_j) \prod_{1\le i\le j\le n-1} (qy_j/x_i;q)_{s_i-1}(qy_i/x_{j+1};q)_{s_{j+1}-1} \\ = \frac{(\prod_{l=1}^n (q;q)_{s_l-1})^n}{(q;q)_{\sum_{l=1}^n s_l-1}} \prod_{1\le i<j\le n} (x_i - x_j)(qx_i/x_j;q)_{s_j-1}(qx_j/x_i)_{s_i-1},\end{aligned} \tag{B.5}$$

and it is also the special case $\nu = \emptyset$ of Okounkov's q-integral formula for the Macdonald polynomials [31]

$$\begin{aligned}\Big(\prod_{1\le i<j\le n}(x_i-x_j)(qx_i/x_j;q)_{k-1}(qx_j/x_i)_{k-1}\Big)^{-1}\sum_y P_\nu(y_1,\dots,y_{n-1};q,t)\prod_{j=1}^{n-1}y_j\prod_{1\le i<j\le n-1}(y_i-y_j) \\ \times \prod_{1\le i\le j\le n-1}(qy_j/x_i;q)_{k-1}(qy_i/x_{j+1};q)_{k-1} = \frac{((q;q)_{k-1})^n}{(q;q)_{kn-1}}\frac{\varepsilon_{t^{n-1},t}(P_\nu)}{\varepsilon_{t^n,t}(P_\nu)}P_\nu(x_1,\dots,x_n;q,t).\end{aligned} \tag{B.6}$$

This latter observation allows us to write (B.6) in a structured form from which a simple derivation in the general ν case follows. Recalling (B.2) and (B.4), and introducing the notation $u_\mu^{(n)}$ ($u_\kappa^{(n-1)}$) from [28, pg. 331] to denote the evaluation map on polynomials in n-variables ($(n-1)$-variables) which sets $x_i = q^{\mu_i}t^{n-i}$ ($y_i = q^{\kappa_i}t^{n-1-i}$), we see that (B.6) can be rewritten

$$\sum_\kappa \frac{u_\kappa^{(n-1)}(P_\nu)}{u_0^{(n-1)}(P_\nu)}\frac{u_0^{(n-1)}(P_\kappa)}{u_0^{(n)}(P_\mu)}\psi_{\mu/\kappa}(q,t)t^{|\kappa|} = \frac{u_\mu^{(n)}(P_\nu)}{u_0^{(n)}(P_\nu)}. \tag{B.7}$$

According to [28, (6.6) pg. 332], the evaluation map satisfies the symmetry relation

$$\frac{u_\kappa^{(n-1)}(P_\nu)}{u_0^{(n-1)}(P_\nu)} = \frac{u_\nu^{(n-1)}(P_\kappa)}{u_0^{(n-1)}(P_\kappa)} \tag{B.8}$$

and so we have

$$\mathrm{LHS(B.7)} = \frac{1}{u_0^{(n)}(P_\mu)}u_\nu^{(n-1)}\Big(\sum_\kappa P_\kappa(y_1,\dots,y_{n-1})\psi_{\mu/\kappa}(q,t)t^{|\kappa|}\Big). \tag{B.9}$$

The fundamental recurrence (3.1) allows the summation over κ in this expression to be performed, leaving us with

$$\mathrm{LHS(B.7)} = \frac{1}{u_0^{(n)}(P_\mu)}u_\nu^{(n-1)}\Big(P_\mu(ty_1,\dots,ty_{n-1},1)\Big). \tag{B.10}$$

But according to the definitions of $u_\nu^{(n-1)}$ and $u_\nu^{(n)}$ we have

$$u_\nu^{(n-1)}\Big(P_\mu(ty_1,\dots,ty_{n-1},1)\Big) = u_\nu^{(n)}(P_\mu).$$

Substituting this in (B.10) then using the symmetry relation (B.8) in the form

$$\frac{u_\nu^{(n)}(P_\mu)}{u_0^{(n)}(P_\mu)} = \frac{u_\mu^{(n)}(P_\nu)}{u_0^{(n)}(P_\nu)}$$

gives the RHS of (B.7).

Appendix C

Matrix Bessel functions

Guhr and Kohler [22] defined the matrix Bessel function

$$\Phi_N^{(\beta)}(x,k) = \int d\mu(U)\,\exp\Big(i\mathrm{Tr}\,U^\dagger x U k\Big)$$

where $U \in U(N;\beta)$ with $U(N;1) = O(N)$ (orthogonal group), $U(N;2) = U(N)$ (unitary group) and $U(N;4) = Sp(N)$ (unitary symplectic group). Furthermore $x = \mathrm{diag}(x_1,\dots,x_N)$, $k = \mathrm{diag}(k_1,\dots,k_N)$ for $\beta = 1,2$ while $x = \mathrm{diag}(x_1,x_1,\dots,x_N,x_N)$, $k = \mathrm{diag}(k_1,k_1\dots,k_N,k_N)$ for $\beta = 4$ (and thus each eigenvalue is doubly degenerate in this case). Let U_N denote the Nth column of U (for $\beta = 4$ each element is itself a 2×2 matrix representing a real quaternion) and construct the random corank 1 projection of x by

$$\Pi x \Pi, \qquad \Pi := \mathbf{1} - U_N U_N^\dagger \tag{C.1}$$

where $\mathbf{1}$ denotes the identity matrix. In [22] the non-zero eigenvalues of this projection are termed the radial Gelfand-Tzeltin coordinates.

Let A denote the $N \times N-1$ matrix with 1's down its diagonal and 0's elsewhere. Then we can write

$$\Phi_N^{(\beta)}(x,k) = \int d\mu(U)\,\exp\Big(i\mathrm{Tr}\,U^\dagger x U k A A^T\Big)\exp\Big(i\mathrm{Tr}\,U^\dagger x U k(\mathbf{1} - AA^T)\Big).$$

Noting that $kA = A\tilde{k}$ with $\tilde{k} = \mathrm{diag}(k_1,\dots,k_{N-1})$ while $k(\mathbf{1} - AA^T) = (\mathbf{1} - AA^T)k_N$, and defining x' as the $(N-1)\times(N-1)$ matrix

$$x' := A^T U^\dagger x U A \tag{C.2}$$

we see that we have

$$\Phi_N^{(\beta)}(x,k) = \int d\mu(U)\,\exp\Big(i\mathrm{Tr}\,ix'\tilde{k}\Big)\exp\Big(ik_N(\mathrm{Tr}\,x - \mathrm{Tr}\,x')\Big).$$

A simple calculation (see [22]) shows that the eigenvalues of (C.2) coincide with the non-zero eigenvalues of (C.1). Consequently the second exponential in the integrand only depends on U_N, so if we decompose $d\mu(U)$ as

$$d\mu(U) = d\mu(\tilde{U})dU_N$$

where $\tilde{U} \in U(N-1;\beta)$ we have the factorization [22]

$$\begin{aligned}\Phi_N^{(\beta)}(x,k) &= \int dU_N\,\exp\Big(ik_N(\mathrm{Tr}\,x - \mathrm{Tr}\,x')\Big)\int d\mu(\tilde{U})\,\exp\Big(i\mathrm{Tr}\,ix'\tilde{k}\Big)\\ &= \int dU_N\,\exp\Big(ik_N(\mathrm{Tr}\,x - \mathrm{Tr}\,x')\Big)\Phi_{N-1}^{(\beta)}(x,k).\end{aligned}$$

Finally, the change of variables from U_N to the eigenvalues of (C.1), with the x's given, is carried out according to Corollary 1.

Appendix D

Sampling from $J_0^{(n,\text{-})}$ in a Monte Carlo calculation

The eigenvalue PDF for Dyson's circular ensembles of random unitary matrices with orthogonal ($\beta = 1$), unitary ($\beta = 2$) and symplectic ($\beta = 4$) symmetry is given by

$$\frac{1}{C_N}\prod_{1\le j<k\le N}|e^{i\theta_k} - e^{i\theta_j}|^\beta. \tag{D.1}$$

x	Exact	Monte Carlo
0.2	0.01687	0.01690
0.4	0.2059	0.207
0.6	0.6641	0.674
0.8	1.1173	1.15
1.0	1.2257	1.30
1.2	1.0208	1.15
1.4	0.8308	1.05

Table 1: The value of $\rho_2(x)$ in the case $\beta = 4$ as computed from (D.4) using $M = 22,500$ distinct samples of γ_β, tabulated against the exact value computed from (D.5). The deterioration of the accuracy of the former as x increases is due in part to amplification of the error caused by the factor of x^β in (D.4).

The corresponding scaled two point correlation function is defined as

$$\rho_2(x) := \lim_{N\to\infty} \frac{N(N-1)}{C_N}|1-e^{2\pi i x/N}|^\beta \\ \times \int_0^{2\pi} d\theta_3 \cdots \int_0^{2\pi} d\theta_N \prod_{l=3}^N |1-e^{i\theta_l}|^\beta |e^{2\pi i x/N} - e^{i\theta_l}|^\beta \prod_{3\le j<k\le N} |e^{i\theta_k} - e^{i\theta_j}|^\beta \tag{D.2}$$

(note that with this scaling the mean eigenvalue spacing is unity). For general even $\beta > 0$ it was shown in [17] that

$$\rho_2(x) = (\beta/2)^\beta \frac{((\beta/2)!)^3}{\beta!(3\beta/2)!}(2\pi x)^\beta \Big\langle \cos 2\pi x(\sum_{j=1}^\beta x_j - \beta/2)\Big\rangle \tag{D.3}$$

where the average is over the PDF $J_0^{(\beta,-)}(x)$ (3.26) with $k = 2/\beta$, $\alpha = \beta = -1 + 2/\beta$. But we know the polynomial $A_\beta^\#(x)$ computed from the random recurrence (4.38) with $k = 2/\beta$, $\alpha_0 = \beta_0 = -1 + 2/\beta$ has zeros with this distribution, and so $\sum_{j=1}^\beta x_j = -[x^{\beta-1}]A_\beta^\#(x) =: \gamma_\beta$ where the symbol $[x^p]$ denotes the coefficient of x^p. According to the general theory of Monte Carlo integration, if we sample γ_β a total of M times we then have

$$\rho_2(x) = (\beta/2)^\beta \frac{((\beta/2)!)^3}{\beta!(3\beta/2)!}(2\pi x)^\beta \Big(\frac{1}{M}\sum_{j=1}^M \cos(2\pi x(\gamma_\beta^{(j)} - \beta/2) + O\Big(\frac{1}{\sqrt{M}}\Big)\Big). \tag{D.4}$$

It is illustrative to compute (D.4) in the case $\beta = 4$, since then we have an alternative formula to (D.3) for the exact evaluation [29],

$$\rho_2(x) = 1 - \Big(\frac{\sin 2\pi x}{2\pi x}\Big)^2 + \frac{1}{2\pi}\frac{d}{dx}\Big(\frac{\sin 2\pi x}{x}\Big)\int_0^{2\pi x} \frac{\sin t}{t}\, dt, \tag{D.5}$$

from which $\rho_2(x)$ can be computed to essentially arbitrary precision, thereby allowing the accuracy of (D.4) to be assertained. The results are listed in Table 1.

Appendix E

Eigenvalue PDF for a generalization of the matrix structure (5.5)

Here we will compute the eigenvalue PDF for the random matrix

$$M = A + XBX^{\dagger} + \sqrt{t}Y \tag{E.1}$$

where A (B) is a fixed $n \times n$ ($m \times m$) Hermitian matrix with distinct eigenvalue $a_1, \dots, a_n$ ($b_1, \dots, b_m$), X is a $n \times m$ complex Gaussian matrix with real and imaginary parts having variance $1/2$ and Y is a random Hermitian matrix from the $n \times n$ GUE (see e.g. [16] for the definition of the latter). Our expression will involve the operator

$$\mathcal{L}^{-1}[f(s)](x) = \lim_{\tau \to 0} \int_{\mathrm{Re}(s)=0} e^{sx+\tau s^2/2} f(s) \, \frac{ds}{2\pi i}, \quad x \in \mathbb{R},$$

which is the inverse of the two sided Laplace transform

$$\mathcal{L}[f(x)](s) = \int_{-\infty}^{\infty} e^{-sx} f(x) \, dx.$$

We will show that in the special case that $t = 0$ and B has rank 1 the term involving $\mathcal{L}^{-1}$ can be evaluated explicitly, reclaiming (5.3) in the case $s_i = 1$ ($i = 1, \dots, n$).

Theorem 6. *The eigenvalue PDF of the random matrix (E.1) is given by*

$$\frac{1}{n!} \det \left[\mathcal{L}^{-1}[e^{ts^2/2} \det(1 + Bs)^{-1}](x_i - a_j) \right]_{i,j=1,\dots,n} \frac{\det[x_i^{j-1}]_{i,j=1,\dots,n}}{\det[a_i^{j-1}]_{i,j=1,\dots,n}}. \tag{E.2}$$

Proof. Clearly, applying a unitary change of basis to A leaves the eigenvalue distribution of M unchanged; we may thus consider the random matrix

$$M' := UAU^{\dagger} + XBX^{\dagger} + \sqrt{t}Y$$

for U Haar-distributed from $U(n)$. Denote by $\mathbf{E}$ the operation of averaging over random variables, and consider the moment generating function (multivariate Laplace transform)

$$f_{M'} : C \to \mathbf{E}(\exp(-\mathrm{Tr}(M'C)))$$

for Hermitian matrices C. We must therefore compute

$$f_{M'}(C) = \mathbf{E}(\exp(-\mathrm{Tr}(UAU^{\dagger}C)))\mathbf{E}(\exp(-\mathrm{Tr}(XBX^{\dagger}C)))\mathbf{E}(\exp(-\mathrm{Tr}(\sqrt{t}YC)))$$

Now, if C has eigenvalues $c_1, \dots, c_n$ then

$$\mathbf{E}(\exp(-\mathrm{Tr}(XBX^{\dagger}C))) = \prod_{i=1}^{m} \prod_{j=1}^{n} \mathbf{E}(\exp(-b_i c_j |X_{ij}|^2)) = \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{1}{1 + b_i c_j} = \prod_{i=1}^{n} \det(1 + Bc_i)^{-1}$$

and

$$\mathbf{E}(\exp(-\mathrm{Tr}((\sqrt{t}YC))) = \mathbf{E}(\exp(-\sum_{i=1}^{n}((\sqrt{t}Y_{ii}c_i)) = \prod_{i=1}^{n} \exp(-tc_i^2/2).$$

Finally, by the Harish-Chandra/Itzykson-Zuber formula (see e.g. [29])

$$\mathbf{E}(\exp(-\mathrm{Tr}(UAU^{\dagger}C))) = (-1)^{n(n-1)/2} \prod_{i=1}^{n} (i-1)! \frac{\det[\exp(-a_j c_i)]_{i,j=1,\dots,n}}{\det[a_i^{j-1}]_{i,j=1,\dots,n} \det[c_i^{j-1}]_{i,j=1,\dots,n}} \tag{E.3}$$

so we have

$$f_{M'}(C) = (-1)^{n(n-1)/2} \prod_{i=1}^{n} (i-1)! \frac{\det[\exp(-a_j c_i)\exp(-tc_i^2/2)\det(1+Bc_i)^{-1}]_{i,j=1,\dots,n}}{\det[a_i^{j-1}]_{i,j=1,\dots,n} \det[c_i^{j-1}]_{i,j=1,\dots,n}}. \tag{E.4}$$

Note that this is analytic for C in a neighbourhood of 0.

On the other hand, suppose we know that a random matrix $\tilde{M}$ has distribution invariant under unitary change of basis, and has eigenvalue PDF

$$\frac{1}{Z} \det[x_i^{j-1}]_{i,j=1,\dots,n} \det[g_j(x_i)]_{i,j=1,\dots,n} \tag{E.5}$$

for some functions g_j with Laplace transform defined in a neighbourhood of 0. Then

$$\begin{aligned}
&\mathbf{E}(\exp(-\mathrm{Tr}(C\tilde{M}))) \\
&\quad = \mathbf{E}(\mathbf{E}_{U \in U(n)}(\exp(-\mathrm{Tr}(CU\tilde{M}U^\dagger)))) \\
&\quad = \frac{(-1)^{n(n-1)/2}\prod_{i=1}^{n}(i-1)!}{Z\det[c_i^{j-1}]_{i,j=1,\dots,n}} \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_n \, \det[\exp(-x_j c_i)]_{i,j=1,\dots,n} \det[g_j(x_i)]_{i,j=1,\dots,n} \\
&\quad = \frac{(-1)^{n(n-1)/2}\prod_{i=1}^{n+1}(i-1)!}{Z\det[c_i^{j-1}]_{i,j=1,\dots,n}} \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_n \, \det\Big[\int_{-\infty}^{\infty} e^{-xc_i} g_j(x)\,dx\Big]_{i,j=1,\dots,n} \\
&\quad = \frac{(-1)^{n(n-1)/2}\prod_{i=1}^{n+1}(i-1)!}{Z\det[c_i^{j-1}]_{i,j=1,\dots,n}} \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_n \, \det[\mathcal{L}[g_j(x)](c_i)]_{i,j=1,\dots,n}
\end{aligned} \tag{E.6}$$

where to obtain the second equality, use has been made of the Harish-Chandra/Itzykson-Zuber formula (E.3). Comparing (E.6) with (E.3) allows us to deduce the value of Z and $g(x)$, which when substituted in (E.5) gives (E.2). □

In the special case $t = 0$, $B = \mathrm{diag}(b, 0, \dots, 0)$, the random matrix in (E.2) coincides with (5.5). In this case

$$\mathcal{L}^{-1}[e^{ts^2/2}\det(1+Bs)^{-1}](x_i - a_j) = \mathcal{L}^{-1}[(1+bs)^{-1}](x_i - a_j) = \chi_{x_i - a_j > 0} \frac{1}{b} e^{-(x_i-a_j)/b}, \quad b > 0$$

and so

$$\det\Big[\mathcal{L}^{-1}[e^{ts^2/2}\det(1+Bs)^{-1}](x_i - a_j)\Big]_{i,j=1,\dots,n} = b^{-n} e^{-\sum_{i=1}^{n}(x_i - a_j)/b} \det[\chi_{x_i \ge a_j}]_{i,j=1,\dots,n}.$$

For $a_1 > a_2 > \cdots > a_n$ the latter determinant is non-zero only when

$$x_1 \ge a_1 \ge x_2 \ge \cdots x_n \ge a_n$$

in which case it is 1. Thus we see that (E.2) reproduces the eigenvalue PDF for the matrices (5.5) in the case that $\vec{x}$ is complex and the eigenvalues of A distinct.

Another special case of interest is $B = 0$. Noting that

$$\mathcal{L}^{-1}[e^{ts^2/2}](x_i - a_j) = (2\pi t)^{-1/2} e^{-(x_i - a_j)^2/2t}$$

we see from Theorem 6 that $A + \sqrt{t}Y$ for Y an element of the GUE has eigenvalue density

$$\frac{1}{n!(2\pi t)^{n/2}} \det[e^{-(x_i-a_j)^2/2t}]_{i,j=1,\dots,n} \frac{\det[x_i^{j-1}]_{i,j=1,\dots,n}}{\det[a_i^{j-1}]_{i,j=1,\dots,n}}.$$

This is a well known consequence of the Harish-Chandra/Itzykson-Zuber formula (see e.g. [29]).

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